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The symmetric, D-invariant and Egorov reductions of the quadrilateral lattice

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Abstract

We present a detailed study of the geometric and algebraic properties of the multidimensional quadrilateral lattice (a lattice whose elementary quadrilaterals are planar; the discrete analog of a conjugate net) and of its basic reductions. To make this study, we introduce the notions of forward and backward data, which allow us to give a geometric meaning to the τ-function of the lattice, defined as the potential connecting these data. Together with the known circular lattice (a lattice whose elementary quadrilaterals can be inscribed in circles; the discrete analog of an orthogonal conjugate net) we introduce and study two other basic and independent reductions of the quadrilateral lattice: the symmetric lattice, for which the forward and backward data coincide, and the d-invariant lattice, characterized by the invariance of a certain natural frame along the main diagonal. We finally discuss the Egorov lattice, which is, at the same time, symmetric, circular and d-invariant. The integrability properties of all these lattices are established using geometric, algebraic and analytic means; in particular, we present a $\bar{\partial}$ formalism to construct large classes of such lattices. We also discuss quadrilateral hyperplane lattices and the interplay between quadrilateral point and hyperplane lattices in all the above reductions.

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1. Introduction

In a recent paper [13] we have introduced the notion of multidimensional quadrilateral lattice (MQL), i.e., a lattice $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{P}^M,\ N \leq M$, with all its elementary quadrilaterals planar, which is the discrete analog of a multidimensional conjugate net [9]. Furthermore, we showed that the planarity constraint (which is a linear constraint) provides a way to construct the lattice uniquely, once a suitable set of initial data is given.

In this paper we present a detailed study of three basic and independent integrable reductions of the quadrilateral lattice: the symmetric lattice, the circular lattice and the d-invariant lattice; we also study the Egorov lattice which is, at the same time, symmetric, circular and d-invariant. All these reductions satisfy additional geometric properties which are compatible with the planarity constraint of the MQL.

The *symmetric* lattice follows from the observation that one can associate, with a given quadrilateral lattice, forward and backward data connected through a potential coinciding with the τ-function of the lattice, and it corresponds to the particular situation in which the backward and forward rotation coefficients coincide. The *circular* lattice, discrete analog of an orthogonal net, is instead characterized by the fact that all its elementary quadrilaterals are inscribed in circles. The *d-invariant* lattice is a MQL characterized by the invariance of a certain natural frame along the main diagonal. The *Egorov* lattice, discrete analog of a Egorov net [2,9], is simultaneously symmetric, circular and d-invariant (for $N = M$), and can be equivalently characterized by the fact that a pair of opposite angles of the elementary quadrilateral consists of right angles.

The geometric properties characterizing the above reductions make use of the connections between point lattices and hyperplane lattices (lattices in the dual space $(\mathbb{P}^M)^*$). In some cases the connection comes from additional structure in the ambient space $\mathbb{P}^M$; in some other cases, it is a consequence of the inner symmetry of the lattice. The precise connections between point and hyperplane lattices corresponding to all the above reductions are also presented in this paper.

Our presentation reflects the effort of constructing a general theory of the MQL and of its reductions and therefore the results will not appear in a chronological order of derivation but rather in a logical order.

Although the research field of integrable discrete geometry is relatively new, the amount of associated results is already very large and it is often difficult to go through the corresponding literature, also because many of these results are not even published, having being presented only during conferences or seminars, or private conversations. A brief but hopefully correct account of the literature closed to the subject considered in this paper is the following.

The proper discrete analog of a conjugate net on a surface was first proposed by Sauer [32]. The MQL equations were first derived by Bogdanov and Konopelchenko [5] as integrable discrete analogs of the Darboux equations for conjugate nets, but without any geometric characterization. The notion of circular lattice was first proposed by Martin et al. [28] and Nutbourne [30] for $N = 2$, $M = 3$, as a discrete analog of surfaces parametrized by curvature lines (see also [4]); later by Bobenko [3] for $N = M = 3$ and, finally, for arbitrary $N \leq M$ by Cieśliński et al. [8]; subsequently, Konopelchenko and Schief [25] have shown that circular lattices in $\mathbb{E}^3$ can be conveniently characterized by solutions of the $(2+1)$-dimensional discrete Sine–Gordon equation [29]. A geometric proof of the integrability of the circular lattice was first given in [8], while the analytic proof of its integrability was given in [16] through the $\bar{\partial}$ method. The notion of Egorov lattice with its right angles characterization was found by Schief [33]. In the derivation of the Egorov lattice, he apparently used the algebraic formulation of the symmetric constraint; this formulation was restricted to the subclass of circular lattices and its geometric meaning was not given [35]. He also found the d-invariance of the Egorov lattice (the Killing vector property) [34]. The finite-gap formulations of the circular and Egorov lattices have also recently appeared in the literature [1,26].

The new results written down in this paper, although already presented in several occasions [14,15,31], are the following:

1. The geometric meaning of the τ-function of the MQL.
2. The theory of integrable hyperplane lattices, and its central role in the reduction theory of MQL.
3. The algebraic and geometric notions of symmetric and d-invariant lattices as basic and independent reductions of the MQL.
4. The successful application of the $\bar{\partial}$ reduction method, already used in the case of circular lattices [16], to all the other reductions.

After this work was completed we were told that the algebraic formulation of a symmetric quadrilateral lattice was already known to Schief [36].

In the rest of this section we summarize the basic results on quadrilateral lattices and the known facts on hyperplanes in projective spaces which will be used in the paper. In Section 2 we introduce the "backward" representation of the quadrilateral lattice and we show that the compatibility between the backward construction and the standard forward construction leads to the existence of a potential which can be identified with the τ-function of the lattice. In Section 3 we first introduce the notion of quadrilateral hyperplane lattice; then we introduce and study the notions of dual, adjoint, conjugate and complementary systems of point and hyperplane lattices. In Section 4 we study the first integrable reduction, the symmetric lattice together with its integrability properties. In Section 5 we discuss, in the same spirit, the second basic reduction, the circular lattice. In Section 6 we define the third basic reduction, the d-invariant lattice and study its properties. Section 7 is devoted to the study of the Egorov lattice which is, at the same time, symmetric, circular and d-invariant. In Section 8 we finally study the integrability properties of all the above lattices from the point of view of their solvability, making use of a $\bar{\partial}$-reduction method recently introduced in [40] in the continuous case and generalized in [16] to a discrete context.

We finally remark that the equations characterizing the above lattices are potentially relevant also in physics, being integrable discretizations of equations arising in hydrodynamics [20,21,24,37] and in quantum field theory [10,17,19].

1.1. Quadrilateral point lattices

Consider a multidimensional quadrilateral lattice, i.e., a mapping $x:\mathbb{Z}^N\to\mathbb{P}^M$, $N\leq M$, with all the elementary quadrilaterals planar [13]. In the affine representation (in which

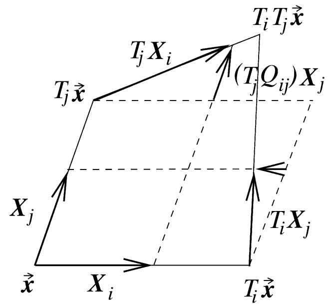

Fig. 1. Definition of the forward data.

the lattice is a mapping $\vec{\boldsymbol{x}} : \mathbb{Z}^N \to \mathbb{R}^M$) the planarity condition can be formulated in terms of the Laplace equations (see also [11])

$$\Delta_i \Delta_j \vec{\boldsymbol{x}} = (T_i A_{ij}) \Delta_i \vec{\boldsymbol{x}} + (T_j A_{ji}) \Delta_j \vec{\boldsymbol{x}}, \quad i \neq j, \ \ i, j = 1, \ldots, N, \tag{1.1}$$

where T_i is the translation operator in the i direction, $\Delta_i = T_i - 1$ and the coefficients A_{ij} satisfy the MQL equation

$$\Delta_k A_{ij} = (T_j A_{jk}) A_{ij} + (T_k A_{kj}) A_{ik} - (T_k A_{ij}) A_{ik}, \quad i \neq j \neq k \neq i. \tag{1.2}$$

It is often convenient to reformulate Eq. (1.1) as a first-order system [13]. We introduce the suitably scaled tangent vectors $\boldsymbol{X}_i$, $i = 1, \ldots, N$,

$$\Delta_i \vec{\boldsymbol{x}} = (T_i H_i) \boldsymbol{X}_i, \tag{1.3}$$

in such a way that the jth variation of $\boldsymbol{X}_i$ is proportional to $\boldsymbol{X}_j$ only (see Fig. 1):

$$\Delta_j \boldsymbol{X}_i = (T_j Q_{ij}) \boldsymbol{X}_j, \quad i \neq j. \tag{1.4}$$

The compatibility condition for the system (1.4) gives the following new form of the MQL equations:

$$\Delta_k Q_{ij} = (T_k Q_{ik}) Q_{kj}, \quad i \neq j \neq k \neq i. \tag{1.5}$$

The scaling factors H_i, called the Lamé coefficients, solve the linear equations

$$\Delta_i H_j = (T_i H_i) Q_{ij}, \quad i \neq j, \tag{1.6}$$

whose compatibility gives Eq. (1.5) again; moreover,

$$A_{ij} = \frac{\Delta_j H_i}{H_i}, \quad i \neq j. \tag{1.7}$$

In [13] it was proven that, given $\frac{1}{2}N(N-1)$ initial quadrilateral surfaces, the quadrilateral lattice $\boldsymbol{x}$ follows uniquely from the planarity constraint. To construct the initial surfaces, one gives N arbitrary intersecting initial curves $\vec{\boldsymbol{x}}_i^{(0)}$, $i = 1, \ldots, N$; the initial (i, j)-surface is then built uniquely assigning the initial data $A_{ij}^{(0)}$, $i \neq j$, as functions of n_i, n_j via

Eq. (1.1). Equivalently, together with the N intersecting initial curves, we can give the initial data $\{H_i^{(0)}, Q_{ij}^{(0)}\}$, meaning that we give the coefficients $H_i^{(0)}$ (or, equivalently, the tangent vectors $\boldsymbol{X}_i^{(0)}$) on the ith initial curve and then the data $Q_{ij}^{(0)}$, $i \neq j$, as functions of n_i, n_j. Therefore, the solution of the MQL equations depends on $N(N-1)$ arbitrary functions of two variables.

Remark. *To make the construction of the lattice possible, in our considerations we assume that we deal with generic lattices, i.e., that the point* x *and its nearest neighbors* $T_1x, \ldots, T_Nx$ *are in general position; in consequence, the subspace* $\langle x, T_1x, \ldots, T_Nx\rangle$ *is a linear subspace of* $\mathbb{P}^M$ *of maximal possible dimension* N.

In this paper we study some distinguished reductions of the MQL which possess additional geometric properties that once imposed on the initial surfaces "propagate" everywhere through the construction of the lattice. Since the quadrilateral lattice is integrable, these reductions will inherit its integrability properties.

In the continuous limit,

$$\Delta_i\vec{\boldsymbol{x}} \sim \varepsilon\frac{\partial}{\partial u_i} = \varepsilon\partial_i, \quad 0 < \varepsilon \ll 1, \tag{1.8}$$

$$Q_{ij} \sim \varepsilon\beta_{ij}, \tag{1.9}$$

the MQL reduces to an N-dimensional conjugate net in $\mathbb{R}^M$ characterized by the Darboux equations [9]:

$$\partial_k\beta_{ij} = \beta_{ik}\beta_{kj}, \quad i \neq j \neq k \neq i. \tag{1.10}$$

1.2. Hyperplanes

In Section 3 we introduce and study the properties of lattices in the dual space, i.e., of hyperplane lattices. These considerations will turn out to be relevant in the reduction theory of the quadrilateral lattices when the introduction of additional geometric structure will allow to establish a direct connection between point lattices and hyperplane lattices.

To make the paper self-contained, in the rest of this section, we summarize some basic known facts on the algebraic representation of the projective space and of its dual.

Points of $\mathbb{P}^M$ are directions (one-dimensional linear subspaces) of $\mathbb{R}^{M+1}$ and they can be represented (up to multiplication by a non-zero scalar factor) by non-zero vectors of $\mathbb{R}^{M+1}$. In a fixed basis $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_M$ of $\mathbb{R}^{M+1}$, the coordinates $\boldsymbol{u} = (u^0, u^1, \ldots, u^M)^{\mathrm{T}}$ of such a vector are called the homogeneous coordinates of the corresponding point $u = [\boldsymbol{u}]$ of the projective space.

The hyperplanes of $\mathbb{P}^M$ are M-dimensional linear subspaces of $\mathbb{R}^{M+1}$ and they can be represented (up to multiplication by a non-zero scalar factor) by non-zero co-vectors of $(\mathbb{R}^{M+1})^* \equiv \mathbb{R}^{M+1}$. The coordinates $\boldsymbol{a}^* = (a_0^*, a_1^*, \ldots, a_M^*)$ of such a co-vector are called the homogeneous coordinates of the corresponding hyperplane $a^* = [\boldsymbol{a}^*]$, and the condition that the point with homogeneous coordinates $\boldsymbol{u} = (u^0, u^1, \ldots, u^M)^{\mathrm{T}}$ belongs to the hyperplane represented by $\boldsymbol{a}^* = (a_0^*, a_1^*, \dots, a_M^*)$ is given by the linear homogeneous equation

$$\langle \boldsymbol{a}^* | \boldsymbol{u} \rangle = a_0^* u^0 + a_1^* u^1 + \cdots + a_M^* u^M = 0. \tag{1.11}$$

Remark (Duality principle). *Notice that Eq.* (1.11) *is "symmetric" in the sense that the expression "the point u belongs to the hyperplane a^*" can be changed into "the hyperplane a^* contains the point u". Geometrically, all hyperplanes* (*points of* $(\mathbb{P}^M)^*$) *passing through a fixed point of $\mathbb{P}^M$ form a hyperplane in $(\mathbb{P}^M)^*$, which is represented by this point, therefore* $((\mathbb{P}^M)^*)^* = \mathbb{P}^M$.

By fixing a hyperplane $\mathbb{P}^{M-1}_\infty$ in $\mathbb{P}^M$, called then the hyperplane at infinity, we can represent the remaining (affine) part $\mathbb{A}^M = \mathbb{P}^M \setminus \mathbb{P}^{M-1}_\infty$ of the projective space by points $\vec{\boldsymbol{v}} \in \mathbb{R}^M$; if the hyperplane at infinity is characterized by $u^0 = 0$, then the points of the affine space can be normalized to $(1, u^1, \dots, u^M)^{\mathrm{T}}$, and $\vec{\boldsymbol{u}} = (u^1, \dots, u^M)^{\mathrm{T}}$.

Hyperplanes in $\mathbb{A}^M$ can be represented (again, up to a non-zero factor) by non-homogeneous linear equations as follows:

$$a_0^* + a_1^* x^1 + \cdots + a_M^* x^M = 0. \tag{1.12}$$

The representation can be made unique by affinization of $(\mathbb{P}^M)^*$, i.e., by removing from $(\mathbb{P}^M)^*$ hyperplanes passing through a fixed point of $\mathbb{P}^M$. For our purposes we assume that this point belongs to $\mathbb{A}^M$, and we identify it with the origin of $\mathbb{R}^M$. Then the equation of any hyperplane which does not pass through the origin, i.e., $a_0^* \neq 0$, can be normalized to have $a_0^* = -1$. Such a hyperplane can be represented by the co-vector $\vec{\boldsymbol{a}}^* \in (\mathbb{R}^M)^*$ and consists of points $\vec{\boldsymbol{x}}$ satisfying the equation

$$\langle \vec{\boldsymbol{a}}^* | \vec{\boldsymbol{x}} \rangle = a_1^* x^1 + \cdots + a_M^* x^M = 1. \tag{1.13}$$

If $\vec{\boldsymbol{v}}$ is a point of the hyperplane represented by $\vec{\boldsymbol{a}}^*$, then the parallel (in the standard sense) hyperplane passing through $t\vec{\boldsymbol{v}}$ is represented by $t^{-1}\vec{\boldsymbol{a}}^*$; equivalently, the equation of such a hyperplane can be written as $\langle \vec{\boldsymbol{a}}^* | \vec{\boldsymbol{x}} \rangle = t$. Taking the limit $t \to \infty$, we infer that the hyperplane at infinity $\mathbb{P}^{M-1}_\infty$ is represented by the zero co-vector $\vec{\boldsymbol{0}}^*$. On the other hand, all the hyperplanes passing through $\vec{\boldsymbol{0}} \in \mathbb{R}^M$ are represented by "infinite" co-vectors; equivalently, the equation of the hyperplane passing through $\vec{\boldsymbol{0}}$ and parallel to that represented by $\vec{\boldsymbol{a}}^*$ can be written as $\langle \vec{\boldsymbol{a}}^* | \vec{\boldsymbol{x}} \rangle = 0$.

Given two hyperplanes a^* and b^* represented by the co-vectors $\vec{\boldsymbol{a}}^*$ and $\vec{\boldsymbol{b}}^*$, the equation of the unique hyperplane passing through the origin and containing their intersection $a^* \cap b^*$ is

$$\langle \vec{\boldsymbol{a}}^* - \vec{\boldsymbol{b}}^* | \vec{\boldsymbol{x}} \rangle = 0. \tag{1.14}$$

Definition 1.1. Two subspaces of co-dimension 2 are called "co-parallel" if there exists a hyperplane passing through $\vec{\boldsymbol{0}}$ and containing them.

Remark. *The above notion is dual to the parallelism of two lines in the affine space.*

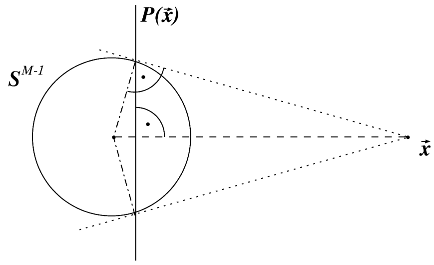

Fig. 2. Polarity with respect to a sphere.

Corollary 1.2. *Two co-dimension* 2 *subspaces obtained by intersection of two pairs of hyperplanes* $a_i^* \cap b_i^*,\ i = 1, 2$, *are co-parallel if the corresponding co-vectors* $\vec{\boldsymbol{a}}_i^* - \vec{\boldsymbol{b}}_i^*,\ i = 1, 2$, *are proportional*.

A *correlation* is a projective mapping between a projective space and its dual

$$\mathcal{C} : \mathbb{P}^M \to (\mathbb{P}^M)^*.$$

In the homogeneous description, such a mapping is given by a linear mapping (given uniquely up to a non-zero scalar factor) between the vector space $\mathbb{R}^{M+1}$ and its dual; if $a^* = [\boldsymbol{a}^*]$, $v = [\boldsymbol{v}]$, and $a^* = \mathcal{C}(v)$, then the correlation $\mathcal{C}$ is represented by a matrix $\boldsymbol{C}$ such that $\boldsymbol{a}^* = (\boldsymbol{C}\boldsymbol{v})^{\mathrm{T}}$.

Any correlation $\mathcal{C}$ defines its adjoint correlation

$$\mathcal{C}^* : ((\mathbb{P}^M)^*)^* = \mathbb{P}^M \to (\mathbb{P}^M)^*$$

being represented by the matrix $\boldsymbol{C}^{\mathrm{T}}$ transposed of $\boldsymbol{C}$. An important class of correlations is provided by *involutory correlations*, i.e., correlations identical to their adjoints. Matrices of such correlations must satisfy the condition that $\boldsymbol{C}^{\mathrm{T}} = \pm\boldsymbol{C}$.

When the matrix of the correlation is symmetric, then the correlation is called *polarity*; we denote it by $\mathcal{P}$. The image $\mathcal{P}(v)$ of a point $v = [\boldsymbol{v}] \in \mathbb{P}^M$ is called the polar hyperplane of v; it consists of points $x = [\boldsymbol{x}]$ satisfying equation $\langle \boldsymbol{P}\boldsymbol{v}|\boldsymbol{x}\rangle = 0$.

Any polarization $\mathcal{P}$ defines the corresponding quadric hypersurface $Q_{\mathcal{P}}$, which consists of points belonging to their polar hyperplanes: $x \in \mathcal{P}(x)$; in the homogeneous description, the quadric is given by equation $\langle \boldsymbol{P}\boldsymbol{x}|\boldsymbol{x}\rangle = 0$.

Example 1.3. Consider the polarization whose quadric is the standard sphere of radius 1 centered at the origin:

$$Q_{\mathcal{P}} = S^{M-1} = \{\vec{\boldsymbol{x}} \in \mathbb{E}^M | \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{x}} = 1\}.$$

Then the polar hyperplane of a point $\vec{\boldsymbol{v}}$ is the hyperplane orthogonal to $\vec{\boldsymbol{v}}$ and passing through the point $\vec{\boldsymbol{v}}/(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}})$ (see Fig. 2). The polar of the origin is the hyperplane at infinity, therefore this polarization is an affine mapping, i.e., it maps parallel lines into co-parallel subspaces (of co-dimension 2).

2. The backward representation of the quadrilateral lattice

In this section we define the backward data $\tilde{\boldsymbol{X}}_i,\ \tilde{H}_i,\ \tilde{Q}_{ij}$ of the quadrilateral lattice. It turns out that the relation between the standard forward data $\boldsymbol{X}_i,\ H_i,\ Q_{ij}$ and the backward data is given in terms of the τ-function, which is one of central objects of the soliton theory.

The backward tangent vectors $\tilde{\boldsymbol{X}}_i$ and the backward Lamé coefficients $\tilde{H}_i,\ i=1,\ldots,N$ are defined with the help of the backward difference operator $\tilde{\Delta}_i := 1 - T_i^{-1}$:

$$\tilde{\Delta}_i\vec{\boldsymbol{x}} = (T_i^{-1}\tilde{H}_i)\tilde{\boldsymbol{X}}_i, \quad \text{or} \quad \Delta_i\vec{\boldsymbol{x}} = \tilde{H}_i(T_i\tilde{\boldsymbol{X}}_i), \tag{2.1}$$

the backward Lamé coefficients are again chosen in such a way (see Fig. 3) that the $\tilde{\Delta}_i$ variation of $\tilde{\boldsymbol{X}}_j$ is proportional to $\tilde{\boldsymbol{X}}_i$ only. We define the backward rotation coefficients $\tilde{Q}_{ij}$ as the corresponding proportionality factors

$$\tilde{\Delta}_i\tilde{\boldsymbol{X}}_j = (T_i^{-1}\tilde{Q}_{ij})\tilde{\boldsymbol{X}}_i, \quad \text{or} \quad \Delta_i\tilde{\boldsymbol{X}}_j = (T_i\tilde{\boldsymbol{X}}_i)\tilde{Q}_{ij}, \quad i \neq j. \tag{2.2}$$

Comparing Eqs. (1.6) and (2.2) we see immediately that the new functions $\tilde{Q}_{ij}$ satisfy the MQL equations (1.5) as well. Moreover, the new scaling factors $\tilde{H}_i$ satisfy the following system of linear equations:

$$\Delta_j\tilde{H}_i = (T_j\tilde{Q}_{ij})\tilde{H}_j, \quad i \neq j, \tag{2.3}$$

whose compatibility condition gives again the MQL equations (1.5).

An easy consequence of Eqs. (2.1)–(2.3) is the following, obvious from a geometric point of view observation.

Proposition 2.1. *The vector function $\vec{\boldsymbol{x}} : \mathbb{Z}^N \to \mathbb{R}^M$ representing a quadrilateral lattice satisfies the backward Laplace equation*

$$\tilde{\Delta}_i\tilde{\Delta}_j\vec{\boldsymbol{x}} = (T_i^{-1}\tilde{A}_{ij})\tilde{\Delta}_i\vec{\boldsymbol{x}} + (T_j^{-1}\tilde{A}_{ji})\tilde{\Delta}_j\vec{\boldsymbol{x}}, \quad i \neq j, \tag{2.4}$$

where, in the notation of this section

$$\tilde{A}_{ij} = \frac{\tilde{\Delta}_j\tilde{H}_i}{\tilde{H}_i}. \tag{2.5}$$

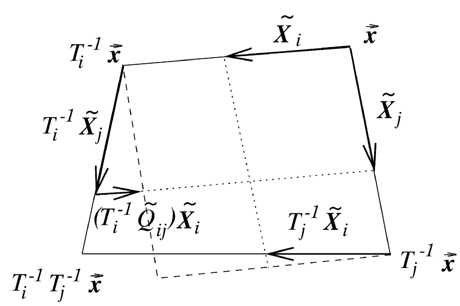

Fig. 3. Definition of the backward data.

The forward and backward rotation coefficients Q_{ij} and $\tilde{Q}_{ij}$ describe the same lattice $\vec{\boldsymbol{x}}$ from different points of view, therefore one can expect their interrelation. Indeed, defining the functions $\rho_i : \mathbb{Z}^N \to \mathbb{R}$ as the proportionality factors between $\boldsymbol{X}_i$ and $T_i\tilde{\boldsymbol{X}}_i$ (both vectors are proportional to $\Delta_i\vec{\boldsymbol{x}}$):

$$\boldsymbol{X}_i = -\rho_i(T_i\tilde{\boldsymbol{X}}_i), \quad T_iH_i = -\frac{1}{\rho_i}\tilde{H}_i, \qquad i = 1, \dots, N, \tag{2.6}$$

we have the following proposition.

Proposition 2.2. *The forward and backward data of the lattice $\vec{\boldsymbol{x}}$ are related through the following formulas:*

$$\rho_j T_j \tilde{Q}_{ij} = \rho_i T_i Q_{ji}, \tag{2.7}$$

and the factors ρ_i are first potentials satisfying equations

$$\frac{T_j\rho_i}{\rho_i} = 1 - (T_iQ_{ji})(T_jQ_{ij}), \quad i \neq j. \tag{2.8}$$

Proof. Using Eqs. (1.4), (2.2) and (2.6), we obtain

$$\boldsymbol{X}_i = -\rho_i T_i\tilde{\boldsymbol{X}}_i = \frac{\rho_i}{T_j\rho_i}(1 - (T_i\tilde{Q}_{ji})(T_j\tilde{Q}_{ij}))(\boldsymbol{X}_i + (T_jQ_{ij})\boldsymbol{X}_j) - \frac{\rho_i}{\rho_j}(T_i\tilde{Q}_{ji})\boldsymbol{X}_j,$$

which, by comparing coefficients in front of the vectors $\boldsymbol{X}_i$, $\boldsymbol{X}_j$, leads to Eqs. (2.7) and (2.8). □

Remark. *Since Q_{ij} and $\tilde{Q}_{ij}$ are both solutions of the MQL equations* (1.5), *then Eqs.* (2.6)–(2.8) *describe a special symmetry transformation of Eq.* (1.5), *first found in* [25] *without any associated geometric meaning*.

The RHS of Eq. (2.8) is symmetric with respect to the interchange of i and j, which implies the existence of a potential $\tau : \mathbb{Z}^N \to \mathbb{R}$, such that

$$\rho_i = \frac{T_i\tau}{\tau}, \tag{2.9}$$

therefore, Eq. (2.8) defines the second potential τ:

$$\frac{(T_iT_j\tau)\tau}{(T_i\tau)(T_j\tau)} = 1 - (T_iQ_{ji})(T_jQ_{ij}), \quad i \neq j. \tag{2.10}$$

The potential τ connecting the forward and backward data

$$T_j(\tau\tilde{Q}_{ij}) = T_i(\tau Q_{ji}), \tag{2.11}$$

$$T_i(\tau\tilde{\boldsymbol{X}}_i) = \tau\boldsymbol{X}_i, \tag{2.12}$$

$$\tau\tilde{H}_i = T_i(\tau H_i) \tag{2.13}$$

is the famous τ-*function* of the quadrilateral lattice.

Corollary 2.3 (The τ-function representation of the MQL equations). *Define* τ_{ij} *by*

$$\tau_{ij} = \tau Q_{ij}, \tag{2.14}$$

then Eq. (2.8) *can be rewritten as*

$$(T_i T_j \tau)\tau = (T_i \tau) T_j \tau - (T_i \tau_{ji}) T_j \tau_{ij}, \tag{2.15}$$

and the MQL equations (1.5) *take the form*

$$(T_k \tau_{ij})\tau = (T_k \tau)\tau_{ij} + (T_k \tau_{ik})\tau_{kj}. \tag{2.16}$$

Remark. *The τ-function representation of the MQL equations was found in* [17] *using the Miwa transformation of the τ-function representation of the Darboux equations*.

We notice that, for a given lattice $\vec{\boldsymbol{x}}$, the forward data $\{\boldsymbol{X}_i, Q_{ij}\}$ are defined up to rescaling by functions $a_i(n_i)$,

$$\boldsymbol{X}_i \to a_i \boldsymbol{X}_i, \qquad T_i H_i \to \frac{1}{a_i} T_i H_i, \qquad T_j Q_{ij} \to \frac{a_i}{a_j} T_j Q_{ij}, \tag{2.17}$$

expressing the freedom in the definition of the vectors $\boldsymbol{X}_i^{(0)}$ on the initial curves. An analogous freedom exists for the backward data

$$T_i \tilde{\boldsymbol{X}}_i \to \frac{1}{b_i} T_i \tilde{\boldsymbol{X}}_i, \qquad \tilde{H}_i \to b_i \tilde{H}_i, \qquad T_j \tilde{Q}_{ij} \to \frac{b_i}{b_j} T_j \tilde{Q}_{ij}. \tag{2.18}$$

The corresponding rescaling of ρ_i and τ is, therefore, given by

$$\rho_i \to a_i b_i \rho_i, \qquad \tau \to \tau \prod_{i=1}^{N} c_i(n_i), \tag{2.19}$$

where

$$\frac{T_i c_i}{c_i} = a_i b_i. \tag{2.20}$$

Finally, we remark that the product $(T_i Q_{ji})(T_j Q_{ij})$, which appears in the definition of the τ-function, is the ratio of the areas of the two affine parallelograms $P(\Delta_i \boldsymbol{X}_j, \Delta_j \boldsymbol{X}_i)$ and $P(\boldsymbol{X}_i, \boldsymbol{X}_j)$ (see Fig. 4).

Unlike the definitions of the forward and backward rotation coefficients, this product is invariant with respect to their possible redefinitions given by Eq. (2.17). It can be seen expressing the product, using Eq. (1.7), in terms of the data A_{ij} as follows:

$$(T_i Q_{ji})(T_j Q_{ij}) = \frac{(T_i A_{ij})(T_j A_{ji})}{(T_i A_{ij} + 1)(T_j A_{ji} + 1)}. \tag{2.21}$$

Observe finally that Eq. (2.7) leads to

$$(T_i Q_{ji})(T_j Q_{ij}) = (T_i \tilde{Q}_{ji})(T_j \tilde{Q}_{ij}), \tag{2.22}$$

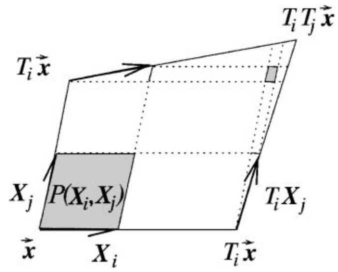

Fig. 4. Areas of two parallelograms.

which implies that the discussed product quantity is also the ratio of the areas of the backward parallelograms $P(\tilde{\Delta}_i T_i T_j \tilde{\boldsymbol{X}}_j, \tilde{\Delta}_j T_i T_j \tilde{\boldsymbol{X}}_i)$ and $P(T_i T_j \tilde{\boldsymbol{X}}_j, T_i T_j \tilde{\boldsymbol{X}}_i)$.

3. Hyperplane lattices

Consider a lattice $\boldsymbol{y}^* : \mathbb{Z}^N \to (\mathbb{P}^M)^*$, $N \leq M$, in the space of hyperplanes of $\mathbb{P}^M$, which we call the *hyperplane lattice*. The space $(\mathbb{P}^M)^*$, called also dual space to $\mathbb{P}^M$, has a natural projective structure and a priori one expects that the algebraic description of the quadrilateral lattices in the dual space be the same like that of quadrilateral point lattices and, therefore, the considerations of the previous sections can be applied to hyperplane lattices without essential modifications. However, this section is devoted to investigate hyperplane lattices from a geometric point of view and to make clear the geometric content of their algebraic description.

3.1. Quadrilateral hyperplane lattices

The basic property of quadrilateral lattices, i.e., the planarity of their elementary quadrilaterals, when applied to hyperplane lattices, can be formulated as follows.

Definition 3.1. The hyperplane lattice $\boldsymbol{y}^* : \mathbb{Z}^N \to (\mathbb{P}^M)^*$ is quadrilateral if, for any $i, j = 1, \ldots, N$, $i \neq j$, the hyperplane $T_i T_j y^*$ contains the subspace $y^* \cap T_i y^* \cap T_j y^*$.

To explain this definition notice that if the hyperplane lattice is given in homogeneous coordinates by the function $\boldsymbol{y}^* : \mathbb{Z}^N \to (\mathbb{R}^{M+1})^* \setminus \{\boldsymbol{0}^*\}$, then Definition 3.1 states that the four co-vectors $T_i T_j \boldsymbol{y}^*$, $T_i \boldsymbol{y}^*$, $T_j \boldsymbol{y}^*$, and $\boldsymbol{y}^*$ are linearly dependent. If $T_i \boldsymbol{y}^*$, $T_j \boldsymbol{y}^*$, $\boldsymbol{y}^*$ are linearly independent, then the co-vector $T_i T_j \boldsymbol{y}^*$ representing the hyperplane $T_i T_j y^*$ is a linear combination of the co-vectors $\boldsymbol{y}^*$, $T_i \boldsymbol{y}^*$ and $T_j \boldsymbol{y}^*$,

$$T_i T_j \boldsymbol{y}^* = \alpha T_i \boldsymbol{y}^* + \beta T_j \boldsymbol{y}^* + \gamma \boldsymbol{y}^*. \tag{3.1}$$

This equation can be transformed into the Laplace equation

$$\Delta_i \Delta_j \boldsymbol{y}^* = (T_i A^*_{ij}) \Delta_i \boldsymbol{y}^* + (T_j A^*_{ji}) \Delta_j \boldsymbol{y}^* + C^*_{(ij)} \boldsymbol{y}^*, \quad i \neq j. \tag{3.2}$$

In the affine gauge, the coefficients $\alpha,\ \beta$ and γ of the decomposition (3.1) are subjected to the constraint

$$\alpha+\beta+\gamma=1, \tag{3.3}$$

and Eq. (3.2) reduces to

$$\Delta_i\Delta_j\vec{\boldsymbol{y}}^*=(T_iA^*_{ij})\Delta_i\vec{\boldsymbol{y}}^*+(T_jA^*_{ji})\Delta_j\vec{\boldsymbol{y}}^*,\quad i\neq j. \tag{3.4}$$

Remark. *In our considerations we always assume we deal with generic lattices, i.e., that the hyperplane y^* and its forward neighbors $T_1y^*,\ldots,T_Ny^*$ (and backward neighbors $T_1^{-1}y^*,\ldots,T_N^{-1}y^*$) are in general position, i.e., their equations are linearly independent. In consequence, the intersection $y^*\cap T_1y^*\cap\cdots\cap T_Ny^*$ (and $y^*\cap T_1^{-1}y^*\cap\cdots\cap T_N^{-1}y^*$) is a linear subspace of $\mathbb{P}^M$ of co-dimension N (of dimension $M-N$).*

Example 3.2. Given a two-dimensional quadrilateral lattice x in the three-dimensional projective space, define the lattice y^* of the hyperplanes passing through $x,\ T_1x$ and T_2x. Because of the planarity of the elementary quadrilaterals of x, it is easy to see that the four hyperplanes $y^*,\ T_1y^*,\ T_2y^*$ and $T_1T_2y^*$ intersect in the point T_1T_2x. Therefore, the (hyper)plane lattice y^* is quadrilateral.

Example 3.3. Correlations map quadrilateral point lattices into quadrilateral hyperplane lattices.

3.2. Dual systems

We first recall that a quadrilateral lattice $\vec{\boldsymbol{x}}'$ is called parallel to the quadrilateral lattice $\vec{\boldsymbol{x}}$ [18] (or obtained from $\vec{\boldsymbol{x}}$ via the Combescure transformation), if the tangents to both lattices are parallel in the corresponding points: $\Delta_i\vec{\boldsymbol{x}}'\sim\Delta_i\vec{\boldsymbol{x}}$. In consequence, the scaled tangent vectors $\boldsymbol{X}_i'$ of the lattice $\vec{\boldsymbol{x}}'$ can be chosen to be equal to those of the lattice $\vec{\boldsymbol{x}}:\boldsymbol{X}_i'=\boldsymbol{X}_i$; then the rotation coefficients of both lattices coincide as well: $Q_{ij}=Q'_{ij}$, and the Lamé coefficients H_i and H_i' are solutions of the same equation.

In this section we will learn how to construct quadrilateral hyperplane lattices using systems of parallel quadrilateral point lattices.

Definition 3.4. Consider a system of M parallel point lattices in $\mathbb{A}^M\ \vec{\boldsymbol{x}}_{(k)},\ k=1,\ldots,M$, whose corresponding vectors are linearly independent. Denote by $\vec{\boldsymbol{y}}^*_{(k)},\ k=1,\ldots,M$, the system of hyperplane lattices uniquely defined by the properties that $\vec{\boldsymbol{y}}^*_{(k)}$ passes through $\vec{\boldsymbol{x}}_{(k)}$ and is spanned by the vectors $\vec{\boldsymbol{x}}_{(l)},\ l\neq k$, i.e.,

$$\langle\vec{\boldsymbol{y}}^*_{(k)}|\vec{\boldsymbol{x}}_{(l)}\rangle=\delta_{kl}. \tag{3.5}$$

We call such a system of hyperplane lattices the *dual system* to the system of parallel point lattices $\vec{\boldsymbol{x}}_{(k)}$.

The aim of this section is to prove that the hyperplane lattices $\vec{\boldsymbol{y}}^*_{(k)}$ are quadrilateral hyperplane lattices.

Definition 3.5. Fix a basis $\{\vec{\boldsymbol{e}}_k\}_{k=1}^M$ in the ambient space $\mathbb{R}^M$ and arrange the parallel system of point lattices in the *matrix* $\boldsymbol{\Omega}$ *of the system*:

$$\boldsymbol{\Omega} = (\vec{\boldsymbol{x}}_{(1)}, \ldots, \vec{\boldsymbol{x}}_{(M)}), \tag{3.6}$$

equivalently, the matrix $\boldsymbol{\Omega}$ represents a linear operator

$$\boldsymbol{\Omega} = \sum_{k=1}^{M} \vec{\boldsymbol{x}}_{(k)} \otimes \vec{\boldsymbol{e}}_k^*,$$

where $\{\vec{\boldsymbol{e}}_k^*\}_{k=1}^M$ is the dual basis of $\{\vec{\boldsymbol{e}}_k\}_{k=1}^M$, i.e., $\langle \vec{\boldsymbol{e}}_k^* | \vec{\boldsymbol{e}}_l \rangle = \delta_{kl}$.

Corollary 3.6. *The components of the dual system* $\vec{\boldsymbol{y}}^*_{(k)}$ *in the basis* $\{\vec{\boldsymbol{e}}_k^*\}_{k=1}^M$ *are given by the rows of the matrix* $\boldsymbol{\Omega}^{-1}$.

Let us arrange the coefficients $H_{i(k)},\ i=1,\ldots,N,\ k=1,\ldots,M$, into the row-vectors

$$\boldsymbol{X}_i^* = (H_{i(1)}, \ldots, H_{i(M)}), \qquad \boldsymbol{X}_i^* = \sum_{k=1}^{M} H_{i(k)} \vec{\boldsymbol{e}}_k^*,$$

then $\boldsymbol{X}_i^*,\ i=1,\ldots,N$, form a (co)vector valued solution of the adjoint linear problem (1.6) and the matrix $\boldsymbol{\Omega}$ can be found from equations

$$\Delta_i \boldsymbol{\Omega} = \boldsymbol{X}_i \otimes (T_i \boldsymbol{X}_i^*). \tag{3.7}$$

It was shown in [27] that the matrix $\boldsymbol{\Omega}$ plays a relevant role in the theory of transformations of quadrilateral lattices.

The following theorem, which contains, as particular cases, all the classical transformations of a quadrilateral lattice [18] was proven in [27].

Theorem 3.7. *Let* $Q_{ij},\ \boldsymbol{X}_i,\ \boldsymbol{X}_i^*$ *and* $\boldsymbol{\Omega}$ *be defined as above; then the following functions*

$$Q'_{ij} = Q_{ij} - \langle \boldsymbol{X}_j^* | \boldsymbol{\Omega}^{-1} | \boldsymbol{X}_i \rangle \tag{3.8}$$

solve the MQL equations, the vectors $\boldsymbol{X}'_i = \boldsymbol{\Omega}^{-1}\boldsymbol{X}_i,\ \boldsymbol{X}_i^{*'} = \boldsymbol{X}_i^*\boldsymbol{\Omega}^{-1}$ *are solutions of the linear systems* (1.4) and (1.6) *for* Q'_{ij}, *and the corresponding potential*

$$\Delta_i \boldsymbol{\Omega}' = \boldsymbol{X}'_i \otimes (T_i \boldsymbol{X}_i^{*'}) \tag{3.9}$$

is given by

$$\boldsymbol{\Omega}' = C - \boldsymbol{\Omega}^{-1}, \tag{3.10}$$

where C *is a constant operator*.

Denote by $\vec{\boldsymbol{x}}^*_{(k)}$ the rows of $\boldsymbol{\Omega}$, then

$$\boldsymbol{\Omega} = \sum_{k=1}^{M} \vec{\boldsymbol{e}}_k \otimes \vec{\boldsymbol{x}}^*_{(k)}.$$

Lemma 3.8. *The rows $\vec{\boldsymbol{x}}^*_{(k)}$ of the matrix $\boldsymbol{\Omega}$ represent a system of co-parallel quadrilateral hyperplane lattices, which we call the adjoint system to $\vec{\boldsymbol{x}}_k$.*

Proof. Let us rewrite Eq. (3.7) in a backward form

$$\tilde{\Delta}_i \boldsymbol{\Omega} = \boldsymbol{X}^*_i \otimes (T_i^{-1} \boldsymbol{X}_i), \tag{3.11}$$

which gives

$$\tilde{\Delta}_i \vec{\boldsymbol{x}}^*_{(k)} = (T_i^{-1} H^*_{i(k)}) \boldsymbol{X}^*_i, \tag{3.12}$$

where $H^*_{i(k)}$ is the kth component of the vector $\boldsymbol{X}_i$. Comparing Eqs. (1.4), (2.1) and (2.3) we infer that the co-vectors $\vec{\boldsymbol{x}}^*_{(k)}$ satisfy the backward Laplace equations

$$\tilde{\Delta}_i \tilde{\Delta}_j \vec{\boldsymbol{x}}^*_{(k)} = (T_i^{-1} \tilde{A}^*_{ij(k)}) \tilde{\Delta}_i \vec{\boldsymbol{x}}^*_{(k)} + (T_j^{-1} \tilde{A}^*_{ji(k)}) \tilde{\Delta}_j \vec{\boldsymbol{x}}^*_{(k)}, \quad i \neq j, \tag{3.13}$$

where

$$\tilde{A}^*_{ij(k)} = \frac{\tilde{\Delta}_j H^*_{i(k)}}{H^*_{i(k)}},$$

and therefore (see Proposition 2.1) also the forward Laplace equations.

Finally, since $\tilde{\Delta}_i \vec{\boldsymbol{x}}^*_{(k)} \sim \tilde{\Delta}_i \vec{\boldsymbol{x}}^*_{(l)}$, then the corresponding co-dimension 2 subspaces $x^*_{(k)} \cap T_i^{-1} x^*_{(k)}$ and $x^*_{(l)} \cap T_i^{-1} x^*_{(l)}$ of hyperplane lattices are co-parallel in the sense of Definition 1.1. □

Remark. *Given the parallel system $\vec{\boldsymbol{x}}_{(k)}$, $k = 1, \ldots, M$, the corresponding adjoint system $\vec{\boldsymbol{x}}^*_{(k)}$ is given up to a fixed basis used to define $\boldsymbol{\Omega}$; on the contrary, the dual system of hyperplane lattices $\vec{\boldsymbol{y}}^*_{(k)}$ is given uniquely.*

Corollary 3.9. *Notice that the forward rotation coefficients of the system $\vec{\boldsymbol{x}}_{(k)}$ are the backward rotation coefficients of the system $\vec{\boldsymbol{x}}^*_{(k)}$: $Q_{ij} = \tilde{Q}^*_{ij}$.*

Combining the above lemma with Theorem 3.7, we get the following theorem.

Theorem 3.10. *The hyperplane lattices $\vec{\boldsymbol{y}}^*_{(k)}$ of the dual system to the system of parallel quadrilateral point lattices $\vec{\boldsymbol{x}}_{(k)}$ are co-parallel quadrilateral hyperplane lattices.*

3.3. The adjoint and conjugate lattices

Definition 3.11. The quadrilateral point lattice $\vec{\boldsymbol{x}} : \mathbb{Z}^N \to \mathbb{A}^M$ and the quadrilateral hyperplane lattice $\vec{\boldsymbol{x}}^* : \mathbb{Z}^N \to (\mathbb{A}^M)^*$ are called adjoint if the forward rotation coefficients of the point lattice are backward rotation coefficients of the hyperplane lattice.

Corollary 3.12. *Equivalently, the forward rotation coefficients of the hyperplane lattice are backward rotation coefficients of its adjoint point lattice.*

Definition 3.13. The point lattice $x : \mathbb{Z}^N \to \mathbb{P}^M$ and the hyperplane lattice $y^* : \mathbb{Z}^N \to (\mathbb{P}^M)^*$ are called conjugate if there exists a one-to-one correspondence between both lattices such that the points x of the point lattice belong to the corresponding hyperplanes y^* of the hyperplane lattice.

Corollary 3.14. *Observe that this notion is self-dual in the sense of the standard duality between* $\mathbb{P}^M = ((\mathbb{P}^M)^*)^*$ *and* $(\mathbb{P}^M)^*$.

Remark. *We are interested only in a situation in which x is a quadrilateral point lattice and y^* is a quadrilateral hyperplane lattice.*

The notion of conjugacy between point lattices and hyperplane lattices is the natural generalization of the notion of conjugacy between point lattices and rectilinear congruences (line lattices with any two neighboring lines coplanar) introduced in [18]. As it was also shown in [18], the lattices parallel to $\vec{\boldsymbol{x}}$ describe transversal congruences conjugate to the quadrilateral point lattice x. Moreover, the tangent congruences can also be obtained in this way via singular limits.

Corollary 3.15. *Given a quadrilateral point lattice x in $\mathbb{P}^M$ and given $(M-1)$ linearly independent congruences conjugate to x, then the hyperplane lattice y^* conjugate to x and spanned by the lines of these congruences is quadrilateral.*

3.4. The complementary lattice

The linear system (1.4) describes the variation of the normalized tangent vectors $\boldsymbol{X}_i$ of a quadrilateral point lattice in directions $j \neq i$, and leads to the MQL equations (1.5). In this section we study the variation of the vectors $\boldsymbol{X}_i$ in the corresponding ith directions of the lattice. Discussion of such variations naturally leads to the definition of a hyperplane lattice, which will be called the *complementary* lattice.

Consider the quadrilateral lattice $\vec{\boldsymbol{x}} : \mathbb{Z}^N \to \mathbb{R}^M$ with the given set of tangent vectors $\boldsymbol{X}_i$, $i = 1, \dots, N$, and the corresponding set of the Lamé and rotation coefficients H_i, Q_{ij}, $i, j = 1, \dots, N$, satisfying Eqs. (1.3)–(1.6). Let us find $M - N$ new solutions Q_{ai}, $a = N+1, \dots, M$, $i = 1, \dots, N$, of the adjoint linear system (1.6), i.e.,

$$\Delta_i Q_{aj} = (T_i Q_{ai}) Q_{ij}, \qquad (3.14)$$

and let us define $M-N$ vectors $\boldsymbol{X}_a,\ a=N+1,\dots,M$, via an analog of Eq. (1.3):

$$\Delta_i \boldsymbol{X}_a = (T_i Q_{ai})\boldsymbol{X}_i. \tag{3.15}$$

Remark. *The vectors $\boldsymbol{X}_a,\ a=N+1,\dots,M$, are the Combescure transforms of the lattice $\vec{\boldsymbol{x}}$, but it was not accidental that we gave to Eqs.* (3.14) and (3.15) *the form of the Darboux equations* (1.5) *and of the linear problem* (1.4).

When the full set of vectors $\boldsymbol{X}_k,\ k=1,\dots,M$, is linearly independent, we obtain, in each point of the lattice $\vec{\boldsymbol{x}}$, a basis of the whole space $\mathbb{R}^M$; this type of basis along a quadrilateral lattice has been considered already in [16] and can be called the *extended basis* along the lattice. By $\tilde{\boldsymbol{Y}}^*_k,\ k=1,\dots,M$, we denote the dual basis in $(\mathbb{R}^M)^*$:

$$\langle \tilde{\boldsymbol{Y}}^*_k | \boldsymbol{X}_\ell \rangle = \delta_{k\ell}, \quad k,\ell = 1,\dots,M. \tag{3.16}$$

The linear system (1.4) describes the decomposition of $T_i\boldsymbol{X}_j,\ i\neq j$; let us decompose $T_i\boldsymbol{X}_i$ in the full basis

$$\Delta_i \boldsymbol{X}_i = \tilde{P}^*_i \boldsymbol{X}_i - \sum_{k\neq i,k=1}^{M} \tilde{P}^*_{ik}\boldsymbol{X}_k. \tag{3.17}$$

We will study properties of the coefficients $\tilde{P}^*_i,\ \tilde{P}^*_{ij}$ and their relation to previously introduced objects.

Proposition 3.16. *The vectors $\tilde{\boldsymbol{Y}}^*_k$ satisfy equations*

$$\Delta_i \tilde{\boldsymbol{Y}}^*_k = (T_i \tilde{\boldsymbol{Y}}^*_i)\tilde{P}^*_{ik}, \quad i\neq k,\ \ i=1,\dots,N,\ \ k=1,\dots,M, \tag{3.18}$$

$$\Delta_i \tilde{\boldsymbol{Y}}^*_i = -(T_i \tilde{\boldsymbol{Y}}^*_i)\tilde{P}^*_i - \sum_{k\neq i,k=1}^{M} (T_i\tilde{\boldsymbol{Y}}^*_k)(T_i Q_{ki}), \quad i=1,\dots,N. \tag{3.19}$$

Proof. Assume a decomposition of $\Delta_i\tilde{\boldsymbol{Y}}^*_\ell$ in the basis $T_i\tilde{\boldsymbol{Y}}^*_\ell,\ \ell=1,\dots,M$,

$$\Delta_i \tilde{\boldsymbol{Y}}^*_\ell = \sum_{k=1}^{M} \Gamma^k_{i\ell}(T_i\tilde{\boldsymbol{Y}}^*_k), \tag{3.20}$$

where

$$\Gamma^k_{i\ell} = \langle \Delta_i \tilde{\boldsymbol{Y}}^*_\ell | T_i \boldsymbol{X}_k\rangle, \quad i=1,\dots,N,\ \ k,\ell=1,\dots,M.$$

Using Eq. (3.16), we obtain that

$$\Gamma^k_{i\ell} = -\langle \tilde{\boldsymbol{Y}}^*_\ell | \Delta_i \boldsymbol{X}_k\rangle, \tag{3.21}$$

which, together with Eqs. (1.4), (3.16)–(3.18), concludes the proof. □

Corollary 3.17. *Eq.* (3.18) *can be split into the standard backward linear problem*

$$\Delta_i \tilde{\boldsymbol{Y}}^*_j = (T_i \tilde{\boldsymbol{Y}}^*_i)\tilde{P}^*_{ij}, \quad i \neq j, \;\; i = 1, \dots, N, \tag{3.22}$$

and the backward linear equations for the supplementary co-vectors

$$\Delta_i \tilde{\boldsymbol{Y}}^*_a = (T_i \tilde{\boldsymbol{Y}}^*_i)\tilde{P}^*_{ia}, \quad i = 1, \dots, N, \;\; a = N+1, \dots, M. \tag{3.23}$$

The compatibility condition of these equations gives the Darboux equations for the backward rotation coefficients $\tilde{P}^*_{ij}$, $i \neq j = 1, \dots, N$,

$$\Delta_k \tilde{P}^*_{ij} = (T_k \tilde{P}^*_{ik})\tilde{P}^*_{kj}, \quad k \neq i, \;\; j = 1, \dots, N, \tag{3.24}$$

and the supplementary backward linear equations

$$\Delta_i \tilde{P}^*_{ja} = (T_i \tilde{P}^*_{ij})\tilde{P}^*_{ia}, \quad i \neq j = 1, \dots, N, \;\; a = N+1, \dots, M. \tag{3.25}$$

Corollary 3.18. *The compatibility of Eqs.* (3.18) *and* (3.19) *gives*

$$\tilde{P}^*_i = T_i Q_{ii} - \tilde{P}^*_{ii}, \quad i = 1, \dots, N, \tag{3.26}$$

where Q_{ii} (*and similarly* $\tilde{P}^*_{ii}$) *are potentials defined in* [16] *for any solution of the MQL system by the equations*

$$\Delta_j Q_{ii} = (T_j Q_{ij}) Q_{ji}, \quad \Delta_j \tilde{P}^*_{ii} = (T_j \tilde{P}^*_{ij})\tilde{P}^*_{ji}, \qquad j \neq i. \tag{3.27}$$

Moreover, from the same compatibility, we obtain the following equation:

$$\Delta_i Q_{ij} + \tilde{\Delta}_j \tilde{P}^*_{ij} - \tilde{P}^*_i Q_{ij} + \tilde{P}^*_{ij}(T_j^{-1}\tilde{P}^*_j) + \sum_{k\neq i,j;k=1}^{M} \tilde{P}^*_{ik} Q_{kj} = 0, \quad i \neq j. \tag{3.28}$$

To make the above considerations symmetric, we define a hyperplane lattice which has the vectors $\tilde{\boldsymbol{Y}}^*_i$, $i = 1, \dots, N$, as normalized backward tangent vectors, and $\tilde{P}^*_{ij}$, $i \neq j = 1, \dots, N$, as backward rotation coefficients.

Definition 3.19. Given the quadrilateral lattice $\vec{\boldsymbol{x}} : \mathbb{Z}^N \to \mathbb{R}^M$ together with its extended frame $\boldsymbol{X}_k$ and its dual $\tilde{\boldsymbol{Y}}_k$, $k = 1, \dots, M$, define the *complementary lattice* of $\vec{\boldsymbol{x}}$ as via solution of the following compatible equations:

$$\Delta_i \vec{\boldsymbol{y}}^* = (T_i \tilde{\boldsymbol{Y}}^*_i)\tilde{F}^*_i, \quad i = 1, \dots, N, \tag{3.29}$$

where $\tilde{F}^*_i$, $i = 1, \dots, N$, is a solution of the system (3.25), interpreted now as the adjoint of the linear system (3.22):

$$\Delta_j \tilde{F}^*_i = (T_j \tilde{P}^*_{ji})\tilde{F}^*_j, \quad i \neq j = 1, \dots, N. \tag{3.30}$$

Remark. *The additional vectors* $\tilde{\boldsymbol{Y}}^*_a$ *and functions* $\tilde{P}^*_{ia}$, $a = N+1, \dots, M$ *play a role similar to that of* $\boldsymbol{X}_a$ *and* Q_{ai}.

By simple calculation one can obtain the following result.

Proposition 3.20. *The functions* $v_k = \langle \vec{\boldsymbol{y}}^* | \boldsymbol{X}_k \rangle$, $k = 1, \ldots, M$, *satisfy equations*

$$\Delta_i v_k = (T_i Q_{ki}) v_i, \quad k \neq i, \tag{3.31}$$

$$\Delta_i v_i = \tilde{F}_i^* + \tilde{P}_i^* v_i - \sum_{k \neq i} \tilde{P}_{ik}^* v_k. \tag{3.32}$$

Similarly, functions $\tilde{v}_k^* = \langle \tilde{\boldsymbol{Y}}_k^* | \vec{\boldsymbol{x}} \rangle$, $k = 1, \ldots, M$ *satisfy equations*

$$\Delta_i \tilde{v}_k^* = (T_i \tilde{v}_i^*) \tilde{P}_{ik}, \quad k \neq i, \tag{3.33}$$

$$\Delta_i \tilde{v}_i^* = (T_i H_i) - \tilde{P}_i^* (T_i \tilde{v}_i^*) - \sum_{k \neq i} (T_i Q_{ki}) \tilde{v}_k^*. \tag{3.34}$$

Finally, we present a theorem which can be proved by simple algebra using formulas of Corollaries 3.17 and 3.18, and which contains a geometric characterization of the complementary lattice.

Theorem 3.21. *Consider the quadrilateral lattice* $\vec{\boldsymbol{x}}$ *with the extended frame* $\boldsymbol{X}_k$, $k = 1, \ldots, M$, *and consider a scalar solution* v_k *of the extended linear system* (3.31). *The hyperplane lattice* $\vec{\boldsymbol{y}}^* = \sum_{k=1}^M v_k \tilde{\boldsymbol{Y}}_k^*$, *whose hyperplanes pass through the M points* $(1/v_k)\boldsymbol{X}_k$, *is a complementary lattice of* $\vec{\boldsymbol{x}}$. *Its backward Lamé coefficients* $\tilde{F}_i^*$, $i = 1, \ldots, N$, *can be obtained via formulas* (3.32).

Remark. *In the continuous limit, for* $N = M = 3$, *and with the identification of planes in* $\mathbb{E}^3$ *as points* (*via polarity*), *our complementary hyperplane lattices reduce to the "systèmes complémentaires d'un système conjugué" considered by Darboux* [9, Chapter III].

4. The symmetric lattice

Definition 4.1. A quadrilateral lattice $\vec{\boldsymbol{x}}$ is *symmetric* iff its forward rotation coefficients are its backward rotation coefficients as well, i.e.,

$$\tilde{Q}_{ij} = Q_{ij}. \tag{4.1}$$

The considerations of Section 2 imply the following characterization.

Proposition 4.2. *A quadrilateral lattice is symmetric iff, for a given set of rotation coefficients* Q_{ij}, *there exists a* τ-*function of the lattice such that*

$$T_i(\tau Q_{ji}) = T_j(\tau Q_{ij}), \quad i \neq j, \tag{4.2}$$

or equivalently, in terms of the corresponding first potentials ρ_i,

$$\rho_i T_i Q_{ji} = \rho_j T_j Q_{ij}. \tag{4.3}$$

Remark. *Due to Eqs.* (2.17)–(2.19), *the above definition is independent of the particular choice of the rotation coefficients* Q_{ij}.

It turns out to the following proposition.

Proposition 4.3. *The symmetric lattice is an integrable reduction of the quadrilateral lattice.*

Proof. Recall that, from a geometric point of view, the integrability of a reduction means that if the reduction condition is satisfied on the initial surfaces, then it must propagate in the construction of the lattice.

As it was shown in [13] the solution Q_{ij} of the MQL equations (1.5) is fixed by the values of the rotation coefficients $Q_{ij}^{(0)}$ on the initial surfaces. Therefore, if $Q_{ij}^{(0)} = \tilde{Q}_{ij}^{(0)}$ on the initial surfaces, then they are equal $Q_{ij} = \tilde{Q}_{ij}$ in the whole lattice, since the backward rotation coefficients $\tilde{Q}_{ij}$ satisfy the same equations as Q_{ij}.

The algebraic content of this result is instead expressed by the following equation:

$$T_k C_{ij}^{\mathrm{S}} = C_{ij}^{\mathrm{S}} + (T_k Q_{jk}) C_{ik}^{\mathrm{S}} - (T_k Q_{ik}) C_{jk}^{\mathrm{S}}, \quad i \neq j \neq k, \tag{4.4}$$

where

$$C_{ij}^{\mathrm{S}} := \rho_i T_i Q_{ji} - \rho_j T_j Q_{ij}. \tag{4.5}$$

Eq. (4.4) is a simple consequence of the MQL equations (1.5) and of Eq. (2.8). Again we see that if the constraint (4.2) is satisfied on the initial surfaces (the RHS of Eq. (4.4) is zero), then it propagates transversally through the whole lattice (the LHS of Eq. (4.4) is zero). □

There exists an interesting geometric characterization of the symmetric lattice, which follows from the interpretation of the condition $\tilde{Q}_{ij} = Q_{ij}$.

Lemma 4.4. *The forward and backward rotation coefficients describing an elementary quadrilateral* $\{\vec{\boldsymbol{x}}, T_i\vec{\boldsymbol{x}}, T_j\vec{\boldsymbol{x}}, T_iT_j\vec{\boldsymbol{x}}\}$ *are equal if and only if the parallelograms* $P(T_i\tilde{\boldsymbol{X}}_i, T_j\tilde{\boldsymbol{X}}_j)$ *and* $P(\Delta_i \boldsymbol{X}_j, \Delta_j \boldsymbol{X}_i)$ *of the quadrilateral are similar.*

Proof. The quadrilateral with the initial vertex is described by the following rotation coefficients: $T_i Q_{ji},\ T_j Q_{ij},\ T_i \tilde{Q}_{ji}$ and $T_j \tilde{Q}_{ij}$ connected by Eq. (2.7). Since

$$\Delta_i \boldsymbol{X}_j = -(T_i Q_{ji}) \rho_i T_i \tilde{\boldsymbol{X}}_i, \tag{4.6}$$

then the parallelograms $P(T_i\tilde{\boldsymbol{X}}_i, T_j\tilde{\boldsymbol{X}}_j)$ and $P(\Delta_i \boldsymbol{X}_j, \Delta_j \boldsymbol{X}_i)$ are similar (see Fig. 5) if and only if

$$\rho_j (T_j Q_{ij}) = \rho_i (T_i Q_{ji}), \tag{4.7}$$

which means, due to (2.7), that the backward and forward Q's are equal. □

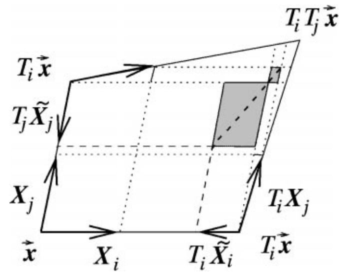

Fig. 5. Similarity of two parallelograms.

Proposition 4.5. *A quadrilateral lattice is symmetric iff, for a given set of the forward tangent vectors $\boldsymbol{X}_i$ of the lattice, there exists a complementary set of the backward tangent vectors $\tilde{\boldsymbol{X}}_i$ such that the parallelograms $P(T_i\tilde{\boldsymbol{X}}_i, T_j\tilde{\boldsymbol{X}}_j)$ and $P(\Delta_i\boldsymbol{X}_j, \Delta_j\boldsymbol{X}_i)$ are similar.*

Remark. *Due to Eqs.* (2.17)–(2.19) *the above characterization of the symmetric lattice is independent of a particular choice of the vectors $\boldsymbol{X}_i$.*

Integrability of the symmetric lattice can be formulated as follows.

Corollary 4.6. *If the system of initial quadrilateral surfaces admits a compatible set of forward–backward data such that $P(T_i\tilde{\boldsymbol{X}}_i, T_j\tilde{\boldsymbol{X}}_j)$ and $P(\Delta_i\boldsymbol{X}_j, \Delta_j\boldsymbol{X}_i)$ are similar, then the similarity of the parallelograms holds in the whole quadrilateral lattice.*

Remark. *Notice that in order to define the symmetric lattice, we need to know what similar parallelograms are.*

The solution of the MQL equations for an N-dimensional symmetric lattice depends on

$$\binom{N}{2}$$

arbitrary functions of two variables, i.e., one-half of the arbitrary functions parametrizing the solution of the MQL equations for generic N-dimensional quadrilateral lattice (see Section 1.1). Given a symmetric lattice equipped with a compatible set of forward and backward data, denote the similarity factor between the parallelograms $P(T_i\tilde{\boldsymbol{X}}_i, T_j\tilde{\boldsymbol{X}}_j)$ and $P(\Delta_i\boldsymbol{X}_j, \Delta_j\boldsymbol{X}_i)$ by $\sigma_{(ij)} = \sigma_{(ji)}$,

$$\Delta_i\boldsymbol{X}_j = \sigma_{(ij)}T_i\tilde{\boldsymbol{X}}_i, \quad \Delta_j\boldsymbol{X}_i = \sigma_{(ij)}T_j\tilde{\boldsymbol{X}}_j, \qquad i \neq j, \tag{4.8}$$

then

$$\sigma_{(ij)} = -\rho_i T_i Q_{ji} = -\rho_j T_j Q_{ij}. \tag{4.9}$$

Therefore, to construct the initial (i, j)-surface of a symmetric lattice, one gives two arbitrary intersecting i-and j-curves and, on them, the tangent vectors $\boldsymbol{X}_i^{(0)}$, $\boldsymbol{X}_j^{(0)}$ and the factors $\rho_i^{(0)}$, $\rho_j^{(0)}$; one finally gives $\sigma_{(ij)} = \sigma_{(ji)}$ as functions of (n_i, n_j).

The descriptions of the symmetric lattice presented above are not explicit. Indeed they involve statements about the existence of suitable potentials. There exists, however, another characterization of the symmetric lattice in terms of the forward rotation coefficients only.

Theorem 4.7. *A quadrilateral lattice is symmetric iff, for different indices i, j, k, its rotation coefficients satisfy the following constraint:*

$$(T_i Q_{ji})(T_j Q_{kj})(T_k Q_{ik}) = (T_j Q_{ij})(T_i Q_{ki})(T_k Q_{jk}). \tag{4.10}$$

In the proof we will use two simple facts (see Eqs. (4.11), (4.12) and (4.15)) valid for a generic quadrilateral lattice.

For a given set of the compatible forward and backward rotation coefficients Q_{ij} and $\tilde{Q}_{ij}$, define functions R_{ij} as

$$R_{ij} = \frac{T_j Q_{ij}}{T_j \tilde{Q}_{ij}}, \tag{4.11}$$

then from Eq. (2.7) it follows that

$$R_{ij} = \frac{1}{R_{ji}}. \tag{4.12}$$

The MQL equations (1.5) can be written as

$$T_k T_j Q_{ij} = T_j Q_{ij} + (T_k T_j Q_{ik})(T_j Q_{kj}), \tag{4.13}$$

which implies

$$T_k T_j Q_{ik} = \frac{T_k T_j Q_{ij} - T_j Q_{ij}}{T_j Q_{kj}}. \tag{4.14}$$

Interchanging the indices j and k in the second equation and eliminating $T_k T_j Q_{ij}$, we obtain

$$\frac{T_k T_j Q_{ik}}{T_k Q_{ik}} = \frac{1 + (T_k Q_{jk})(T_j Q_{ij})/T_k Q_{ik}}{1 - (T_j Q_{kj})(T_k Q_{jk})}. \tag{4.15}$$

Proof. The implication (4.3) ⇒(4.10) is obvious. Let us concentrate on the opposite implication.

Let us start from any set of backward rotation coefficients $\tilde{Q}_{ij}$ related with Q_{ij} via Eqs. (2.6) and (2.7), the condition (4.10) implies

$$\frac{(T_k Q_{jk})(T_j Q_{ij})}{T_k Q_{ik}} = \frac{(T_k \tilde{Q}_{jk})(T_j \tilde{Q}_{ij})}{T_k \tilde{Q}_{ik}}, \tag{4.16}$$

which, together with (4.15) and with the corresponding formula satisfied by the backward rotation coefficients $\tilde{Q}_{ij}$, gives, for j different from i and k,

$$T_j R_{ik} = R_{ik}, \tag{4.17}$$

i.e., R_{ik} is a function of n_i and n_k only. This together with condition (4.10) written in terms of R_{ij} as

$$R_{ij}R_{jk}R_{ki} = 1 \tag{4.18}$$

and with Eq. (4.12) implies the existence of functions $a_i(n_i)$ such that

$$R_{ij}(n_i, n_j) = \frac{a_i(n_i)}{a_j(n_j)}. \tag{4.19}$$

We use the functions a_i to redefine the potentials ρ_i and obtain new backward rotation coefficients $\tilde{Q}_{ij}$ satisfying $Q_{ij} = \tilde{Q}_{ij}$. □

The above characterization of the symmetric lattice works only when the dimension of the lattice is greater than 2. In the following proposition we present an analogous criterion for $N = 2$, which can be useful, e.g., to check directly if the initial quadrilateral surfaces are symmetric.

Proposition 4.8. *A two-dimensional quadrilateral lattice is symmetric iff the function*

$$r_{ij} = \frac{T_j Q_{ij}}{T_i Q_{ji}}, \quad i \neq j, \tag{4.20}$$

satisfies equation

$$\frac{(T_i T_j r_{ij}) r_{ij}}{(T_i r_{ij})(T_j r_{ij})} = \frac{T_i(1 - T_i Q_{ji} T_j Q_{ij})}{T_j(1 - T_i Q_{ji} T_j Q_{ij})}. \tag{4.21}$$

Proof. The implication from (4.3) to (4.21) is trivial. To prove that the condition (4.21) is sufficient, we notice that, in terms of R_{ij}, it can be rewritten as

$$(T_i T_j R_{ij}) R_{ij} = (T_i R_{ij})(T_j R_{ij}), \tag{4.22}$$

which leads again to

$$R_{ij}(n_i, n_j) = \frac{a_i(n_i)}{a_j(n_j)}. \tag{4.23}$$

□

Remark. *In order to check the symmetry condition for the initial surfaces we use the criterion* (4.21) *supplemented by* (4.10) *in the points where the initial surfaces meet.*

As we have anticipated, the constraints discussed in this paper allow one to establish a connection between quadrilateral point lattices and their duals, the quadrilateral hyperplane lattices. The following proposition describes this connection in the case of the symmetry constraint.

Proposition 4.9. *Given a system of parallel quadrilateral lattices* $\{\vec{\boldsymbol{x}}_{(k)}\}_{k=1}^{M}$ *and the associated matrix* $\boldsymbol{\Omega}$ *defined with respect to an orthonormal basis* $\{\vec{\boldsymbol{e}}_k\}_{k=1}^{M}$, $\vec{\boldsymbol{e}}_k \cdot \vec{\boldsymbol{e}}_l = \delta_{kl}$, *then the following properties are equivalent.*

1. *The matrix* $\mathbf{\Omega}$ *of the system is symmetric*:

$$\mathbf{\Omega} = \mathbf{\Omega}^{\mathrm{T}}. \tag{4.24}$$

2. *The polar hyperplane* $\mathcal{P}(\vec{\boldsymbol{x}}_{(k)})$ *of the point lattice* $\vec{\boldsymbol{x}}_{(k)}$ *coincides with the hyperplane lattice* $\vec{\boldsymbol{x}}^{*}_{(k)}$:

$$\mathcal{P}(\vec{\boldsymbol{x}}_{(k)}) = \vec{\boldsymbol{x}}^{*}_{(k)}, \quad k = 1, \ldots, M. \tag{4.25}$$

3. *The lattices* $\vec{\boldsymbol{x}}_{(k)}$, $k = 1, \ldots, M$, *are symmetric. Furthermore, the associated tangent vectors* $\boldsymbol{X}_i$ *and* $\boldsymbol{X}^*_i$ *are related in the following way*:

$$\boldsymbol{X}_i^{\mathrm{T}} = \rho_i (T_i \boldsymbol{X}_i^*), \quad i = 1, \ldots, N. \tag{4.26}$$

Proof. (1) ⇔ (2): The equivalence of (1) and (2) follows immediately from the definitions of the potential matrix Ω and of the polar transformation $\mathcal{P}$.

(1) ⇒ (3). The application of Δ_i to Eq. (4.24) gives the equations

$$\boldsymbol{X}_i \otimes T_i \boldsymbol{X}_i^* = T_i \boldsymbol{X}_i^{*\mathrm{T}} \otimes \boldsymbol{X}_i^{\mathrm{T}}, \tag{4.27}$$

which imply equations $\boldsymbol{X}_i^{\mathrm{T}} = \gamma_i T_i \boldsymbol{X}_i^*$ for some proportionality factor functions γ_i. The linear problem (1.4) and its adjoint (1.6) satisfied by $\boldsymbol{X}_i$ and $\boldsymbol{X}_i^*$ imply that γ_i satisfy Eq. (2.8) (which allows to identify γ_i with ρ_i) and lead to the symmetry condition (4.2).

(3) ⇒ (1): Following a similar strategy, one can show that

$$\Delta_i (\mathbf{\Omega} - \mathbf{\Omega}^{\mathrm{T}}) = 0, \quad i = 1, \ldots, N, \tag{4.28}$$

which implies (4.24) up to some constant of integration. □

Corollary 4.10. *A quadrilateral lattice* $\vec{\boldsymbol{x}}$ *is symmetric iff it is adjoint to its own polar*.

Remark. *In the continuous limit* (1.9), *the symmetric quadrilateral lattice reduces to a symmetric conjugate net, for which the rotation coefficients* β_{ij} *satisfying the Darboux equations* (1.10) *are symmetric:*

$$\beta_{ij} = \beta_{ji}. \tag{4.29}$$

In fact, one should allow for the less restrictive condition

$$\beta_{ij}(u) = \frac{a_i(u_i)}{a_j(u_j)} \beta_{ji}(u), \tag{4.30}$$

which gives (4.29) *after an admissible rescaling of the data*.

The continuous limit of the criterion (4.10)

$$\beta_{ij}\beta_{jk}\beta_{ki} = \beta_{ji}\beta_{kj}\beta_{ik} \tag{4.31}$$

is equivalent to (4.30).

5. The circular lattice

The discrete analog of an N-dimensional orthogonal system of coordinates is the circular lattice.

Definition 5.1. A quadrilateral lattice is circular if and only if any elementary quadrilateral is inscribed in a circle.

An elementary characterization of circular quadrilaterals states that, if a circular quadrilateral is convex, then the sum of its opposite angles is π; when the quadrilateral is skew, then its opposite angles are equal. This leads to a convenient characterization of a circular lattice [16].

Proposition 5.2. *A quadrilateral lattice is circular if and only if*

$$\cos\angle(\boldsymbol{X}_i, T_i\boldsymbol{X}_j) + \cos\angle(\boldsymbol{X}_j, T_j\boldsymbol{X}_i) = 0, \tag{5.1}$$

or equivalently,

$$\boldsymbol{X}_i \cdot T_i\boldsymbol{X}_j + \boldsymbol{X}_j \cdot T_j\boldsymbol{X}_i = 0, \quad i \neq j. \tag{5.2}$$

It turns out to [8,16] give the following propostion.

Proposition 5.3. *The circular lattice is an integrable reduction of the quadrilateral lattice.*

Proof. The proof consists in showing that the circularity property is an admissible constraint for the quadrilateral lattice, i.e., once imposed on the initial surfaces, it propagates transversally through the lattice. This was shown in [8] using purely geometric means. The algebraic proof is instead based on the following formula:

$$T_k C^\circ_{ij} = C^\circ_{ij} + (T_i T_k Q_{jk}) C^\circ_{ik} + (T_j T_k Q_{ik}) C^\circ_{jk}, \quad i \neq j \neq k \neq i, \tag{5.3}$$

where

$$C^\circ_{ij} := \boldsymbol{X}_i \cdot T_i\boldsymbol{X}_j + \boldsymbol{X}_j \cdot T_j\boldsymbol{X}_i, \quad i \neq j, \tag{5.4}$$

which is a direct consequence of Eqs. (1.4) and (1.5). We see that if the circularity constraint (5.2) is satisfied on the initial surfaces (the RHS of (5.3) is zero), then it propagates transversally through the lattice (the LHS of (5.3) is zero). □

Corollary 5.4.
1. *The circularity constraint* (5.2) *implies the following formula* [16] *:*

$$\frac{T_i|\boldsymbol{X}_j|^2}{|\boldsymbol{X}_j|^2} = 1 - (T_i Q_{ji})(T_i Q_{ji}), \tag{5.5}$$

which, compared with Eqs. (2.8)–(2.10), *allows to fix, without loss of generality, the backward formulation of the circular lattice in the following way:*

$$|\boldsymbol{X}_i|^2 = \rho_i = \frac{T_i\tau}{\tau} \Rightarrow |T_i\tilde{\boldsymbol{X}}_i|^2 = \frac{1}{\rho_i} = \frac{\tau}{T_i\tau}. \tag{5.6}$$

2. *The circularity constraint* (5.2), *written in terms of the backward data of the lattice, reads as follows:*

$$\tilde{C}^{\circ}_{ij} := \tilde{\boldsymbol{X}}_i \cdot T_i^{-1}\tilde{\boldsymbol{X}}_j + \tilde{\boldsymbol{X}}_j \cdot T_j^{-1}\tilde{\boldsymbol{X}}_i = 0. \tag{5.7}$$

Proof.
1. Eq. (5.6) is a straightforward consequence of Eq. (5.2) and has been found in [16].
2. Eq. (5.7) follows from the equalities

$$\begin{aligned} C^{\circ}_{ij} &= \rho_i\rho_j(2(T_i\tilde{\boldsymbol{X}}_i)\cdot(T_j\tilde{\boldsymbol{X}}_j) + (T_i\tilde{Q}_{ji})|T_j\tilde{\boldsymbol{X}}_j|^2 + (T_j\tilde{Q}_{ij})|T_i\tilde{\boldsymbol{X}}_i|^2) \\ &= \rho_i\rho_j(1-(T_iQ_{ji})(T_jQ_{ij}))T_iT_j\tilde{C}^{\circ}_{ij}. \end{aligned}$$

The first equality follows from rewriting C°_{ij} in terms of the backward data; the second equality follows from equations

$$T_iT_j\tilde{\boldsymbol{X}}_i = (1-(T_iQ_{ji})(T_jQ_{ij}))^{-1}(T_i\tilde{\boldsymbol{X}}_i + (T_i\tilde{Q}_{ji})T_j\tilde{\boldsymbol{X}}_j), \quad i \neq j, \tag{5.8}$$

which is a straightforward consequence of (2.2). □

Other two convenient characterizations of the circular lattice are contained in the following result found in [25] and explained geometrically in [12].

Proposition 5.5. *A quadrilateral lattice $\vec{\boldsymbol{x}}$ is circular iff the scalars*

$$v_i := (T_i\vec{\boldsymbol{x}} + \vec{\boldsymbol{x}})\cdot\boldsymbol{X}_i, \quad i = 1,\ldots,N \tag{5.9}$$

solve the linear system (1.4) *or, equivalently, iff the function* $|\vec{\boldsymbol{x}}|^2$ (*the square of the norm of* $\vec{\boldsymbol{x}}$) *satisfies the Laplace equation* (1.1) *of* $\vec{\boldsymbol{x}}$.

A distinguished subclass of circular lattices corresponds to the particular case in which the lattice points $\vec{\boldsymbol{x}}$ belong to the sphere of radius $R : |\vec{\boldsymbol{x}}| = R$. In this case there exists, like for the symmetric reduction, an elegant relation between point lattices and hyperplane lattices.

Proposition 5.6. *Given a system of parallel quadrilateral lattices* $\{\vec{\boldsymbol{x}}_{(k)}\}_{k=1}^M$ *and the associated matrix* $\boldsymbol{\Omega}$ *of the system defined with respect to an orthonormal basis* $\{\vec{\boldsymbol{e}}_k\}_{k=1}^M$, *the following properties are equivalent.*
1. *The matrix* $\boldsymbol{\Omega}/R$ *is orthogonal:*

$$\boldsymbol{\Omega}\boldsymbol{\Omega}^{\mathrm{T}} = \boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{\Omega} = R^2\mathbb{I}, \qquad \boldsymbol{\Omega}^{\mathrm{T}} = R^2\boldsymbol{\Omega}^{-1}. \tag{5.10}$$

2. *The polar hyperplane* $\mathcal{P}(\vec{\boldsymbol{x}}_{(k)})$ *coincides with the dual hyperplane* $R^2\vec{\boldsymbol{y}}^*_{(k)}$ *:*

$$\mathcal{P}(\vec{\boldsymbol{x}}_{(k)}) = R^2\vec{\boldsymbol{y}}^*_{(k)}, \quad k = 1, \dots, M. \tag{5.11}$$

3. *The quadrilateral lattices* $\vec{\boldsymbol{x}}_{(k)}/R,\ k = 1, \dots, M$, *form an orthonormal basis:*

$$\vec{\boldsymbol{x}}_{(i)} \cdot \vec{\boldsymbol{x}}_{(j)} = R^2\delta_{ij}, \quad i, j = 1, \dots, M. \tag{5.12}$$

In addition, the associated tangent vectors $\boldsymbol{X}_i,\ \boldsymbol{X}^*_i,\ i = 1, \dots, N$, *are related by the following formulas:*

$$\boldsymbol{X}_i = \frac{\rho_i}{2R^2} T_i(\boldsymbol{\Omega}\boldsymbol{X}_i^{*\mathrm{T}}) = -\frac{\rho_i}{2R^2}\boldsymbol{\Omega}(T_i\boldsymbol{X}_i^{*\mathrm{T}}), \quad i = 1, \dots, N, \tag{5.13}$$

$$T_i\boldsymbol{X}^*_i = -\frac{2}{\rho_i}\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Omega}, \quad i = 1, \dots, N, \tag{5.14}$$

with

$$|\boldsymbol{X}_i|^2 = \rho_i, \quad T_i|\boldsymbol{X}^*_i|^2 = \frac{4R^2}{\rho_i}, \tag{5.15}$$

and satisfy the circularity constraint (5.2) *and its adjoint*

$$C^{\circ *}_{ij} := \boldsymbol{X}^*_i \cdot T_i^{-1}\boldsymbol{X}^*_j + \boldsymbol{X}^*_j \cdot T_j^{-1}\boldsymbol{X}^*_i = 0. \tag{5.16}$$

Proof. The equivalence between (1) and (2) and formula (5.12) is a straightforward consequence of the definitions of $\boldsymbol{\Omega}$, $\vec{\boldsymbol{x}}_{(k)}$ and $\vec{\boldsymbol{y}}^*_{(k)}$. Furthermore, the quadrilateral lattice on a sphere is obviously circular, the circles being the intersections of the sphere with the planes of the elementary quadrilaterals [12].

(1) ⇒ (3). Applying Δ_i to Eq. (5.10) leads to

$$T_i\boldsymbol{X}_i^{*\mathrm{T}} \otimes \boldsymbol{X}_i^{\mathrm{T}} = -R^{-2}\boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{X}_i \otimes T_i(\boldsymbol{X}^*_i\boldsymbol{\Omega}^{\mathrm{T}}), \quad i = 1, \dots, N, \tag{5.17}$$

which implies that

$$\boldsymbol{X}_i = \gamma_i T_i(\boldsymbol{\Omega}\boldsymbol{X}_i^{*\mathrm{T}}), \tag{5.18}$$

$$T_i\boldsymbol{X}^*_i = -\frac{1}{R^2\gamma_i}\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Omega}, \tag{5.19}$$

for some γ_i. Using Eq. (3.7) in (5.18), one obtains

$$\boldsymbol{X}_i = \frac{\gamma_i}{1-\gamma_i|T_i\boldsymbol{X}^*_i|^2}\boldsymbol{\Omega}T_i\boldsymbol{X}_i^{*\mathrm{T}}, \tag{5.20}$$

which together with (5.19) leads to identification of the factors γ_i:

$$\gamma_i = \frac{2}{|T_i\boldsymbol{X}^*_i|^2} = \frac{|\boldsymbol{X}_i|^2}{2R^2}. \tag{5.21}$$

Notice that Eq. (1.6) implies

$$T_i T_j \boldsymbol{X}_i^* = (1 - (T_i Q_{ji})(T_j Q_{ij}))^{-1}(T_i \boldsymbol{X}_i^* + (T_i Q_{ji}) T_j \boldsymbol{X}_j^*), \quad i \neq j. \tag{5.22}$$

Application of the shift in j direction to Eq. (5.20) and using the above identity leads to equations

$$T_j \gamma_i - \gamma_i (1 - (T_i Q_{ji})(T_j Q_{ij})) = 0, \tag{5.23}$$

$$\gamma_i T_i Q_{ji} + \gamma_j T_j Q_{ij} + R^2 \boldsymbol{X}_i \cdot \boldsymbol{X}_j = 0, \tag{5.24}$$

the first of them allows for identification $\rho_i = 2\gamma_i R^2$, while the second gives the circularity condition.

At last, Eqs. (5.13) and (5.15) imply the following relation between the circularity property and its dual:

$$C_{ij}^{\circ} = -\frac{1}{4R^2} \frac{T_i T_j \tau}{\tau} T_i T_j C_{ij}^{\circ *}, \tag{5.25}$$

which implies that also Eq. (5.16) is satisfied. The proof of (3)⇒(1) is similar and is left to the reader. □

Corollary 5.7. *Quadrilateral lattice in a sphere is conjugate to its own polar (with respect to the sphere) hyperplane lattice*.

In the continuous limit, Eq. (5.2) become the orthogonality conditions

$$\boldsymbol{X}_i \cdot \boldsymbol{X}_j = 0, \quad i \neq j, \tag{5.26}$$

and the circular lattice reduces to an orthogonal conjugate net.

6. d-invariant lattice

In this section we introduce and discuss a basic dimensional reduction of the quadrilateral lattice, the *d-invariant lattice*, characterized by the invariance of a certain natural frame along the main diagonal of the lattice.

To do so, it is convenient to put this reduction in the natural framework of the theory of transformations of the quadrilateral lattice discussed in great detail in [18].

From a quadrilateral lattice $\vec{\boldsymbol{x}} : \mathbb{Z}^N \to \mathbb{R}^M$, one can easily construct a new quadrilateral lattice just translating $\vec{\boldsymbol{x}}$ in some coordinate direction and combining this translation with a Combescure transformation. If the translation takes place along the main diagonal, one obtains the new quadrilateral lattice

$$\hat{\boldsymbol{x}} = \mathcal{C}(T\vec{\boldsymbol{x}}), \tag{6.1}$$

where $T := \prod_{i=1}^N T_i$ is the total translation along the main diagonal and $\mathcal{C}(\cdot)$ is the Combescure transformation [18]. From the above definition it follows that

$$\Delta_i \hat{\boldsymbol{x}} = (T_i \hat{H}_i) \hat{\boldsymbol{X}}_i, \tag{6.2}$$

where

$$\hat{\boldsymbol{X}}_i = T\boldsymbol{X}_i \quad (\Rightarrow \hat{Q}_{ij} = TQ_{ij}), \tag{6.3}$$

and $\hat{H}_i$ are solutions of

$$\Delta_j \hat{H}_i = (T_j \hat{H}_j)\hat{Q}_{ji}, \quad i \neq j, \tag{6.4}$$

different from TH_i.

To establish relations between quadrilateral lattices $\vec{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}$, one uses the following relations valid for generic quadrilateral lattices.

Lemma 6.1. *For any subset* $L = \{i_1, \dots, i_L\}$ *of the indices* $1, 2, \dots, N$, *let us define the partial shift* $T_L = \prod_{\ell=1}^{L} T_{i_\ell}$, *then*

$$T_L\boldsymbol{X}_i = \begin{cases} \boldsymbol{X}_i + \sum_{\ell\in L}(T_L Q_{i\ell})\boldsymbol{X}_\ell & \text{if } i \notin L, \\ T_i\boldsymbol{X}_i - (T_i Q_{ii})\boldsymbol{X}_i + \sum_{\ell\in L}(T_L Q_{i\ell})\boldsymbol{X}_\ell & \text{if } i \in L, \end{cases} \tag{6.5}$$

where Q_{ii} *was defined in* (3.27).

Proof. We first prove by induction the case $i \notin L$. For $|L| = 1$ the statement follows from the linear problem (1.4). When $k \notin L$ and $k \neq i$ and the upper part of the formula (6.5) holds, then

$$\begin{aligned} T_{L\cup\{k\}}\boldsymbol{X}_i &= T_L(\boldsymbol{X}_i + (T_k Q_{ik})\boldsymbol{X}_k) \\ &= \boldsymbol{X}_i + (T_{L\cup\{k\}} Q_{ik})\boldsymbol{X}_k + \sum_{\ell\in L} T_L(Q_{i\ell} + (T_k Q_{ik})Q_{k\ell})\boldsymbol{X}_\ell, \end{aligned}$$

and application of the Darboux equations (1.5) concludes the first part of the proof. Notice that applying the shifts T_L and T_k in different order, we obtain the following generalized Darboux equations

$$T_L Q_{ik} = Q_{ik} + \sum_{\ell\in L}(T_L Q_{i\ell})Q_{\ell k}, \quad i \neq k \notin L. \tag{6.6}$$

To show the lower part of the formula (6.5) let us apply the shift T_i to the upper part of it obtaining

$$T_{L\cup\{i\}}\boldsymbol{X}_i = T_i\boldsymbol{X}_i + \sum_{\ell\in L}(T_{L\cup\{i\}} Q_{i\ell})\boldsymbol{X}_\ell + \left(T_i \sum_L (T_L Q_{i\ell})Q_{\ell i}\right)\boldsymbol{X}_i. \tag{6.7}$$

It remains to prove that for a generic lattice and $i \notin L$,

$$T_L Q_{ii} = Q_{ii} + \sum_L (T_L Q_{i\ell})Q_{\ell i}, \tag{6.8}$$

which can be done, again, by simple induction with the help of Eq. (6.6). □

The quadrilateral lattice $\hat{\boldsymbol{x}}$ is characterized by the following property.

Proposition 6.2. *Let $\vec{\boldsymbol{x}}:\mathbb{Z}^N\to\mathbb{R}^M$ be a quadrilateral lattice and let $\hat{\boldsymbol{x}}:\mathbb{Z}^N\to\mathbb{R}^M$ be its transformed quadrilateral lattice* (6.1). *Then*

$$\hat{\boldsymbol{X}}_i = T\boldsymbol{X}_i = T_i\boldsymbol{X}_i - (T_iQ_{ii})\boldsymbol{X}_i + \sum_{\ell=1}^{N}\hat{Q}_{i\ell}\boldsymbol{X}_\ell, \tag{6.9}$$

and consequently,

$$\Delta_i Q_{ij} + \tilde{\Delta}_j\hat{Q}_{ij} - Q_{ij}(T_iQ_{ii} - \hat{Q}_{ii} - 1) - \hat{Q}_{ij}(T_j^{-1}\hat{Q}_{jj} - Q_{jj} + 1) + \sum_{\ell=1,\ell\neq i,j}^{N}\hat{Q}_{i\ell}Q_{\ell j} = 0, \quad i\neq j. \tag{6.10}$$

Proof. Eq. (6.9) follows from Lemma 6.1 for $L=\{1,\dots,N\}$ and from (6.3). Eq. (6.10) is the compatibility condition of Eqs. (1.4) and (6.9). □

The fixed point of transformation (6.1) (and therefore an integrable reduction of the quadrilateral lattice) is represented by the lattices $\vec{\boldsymbol{x}}$ which are parallel to their translations $T\vec{\boldsymbol{x}}:\vec{\boldsymbol{x}}=\mathcal{C}(T\vec{\boldsymbol{x}})$ or, equivalently, for which $T\boldsymbol{X}_i=\boldsymbol{X}_i$.

Definition 6.3. A quadrilateral lattice $\vec{\boldsymbol{x}}:\mathbb{Z}^N\to\mathbb{R}^M$ is *diagonally invariant (d-invariant)* iff

$$T\boldsymbol{X}_i = \boldsymbol{X}_i. \tag{6.11}$$

Remark. *Eq.* (6.11) *implies that*

$$TQ_{ij} = Q_{ij}. \tag{6.12}$$

Remark. *The d-invariant lattice can be described effectively by $N-1$ parameters since it depends on the differences of the variables n_i*:

$$\boldsymbol{X}_i = \boldsymbol{X}_i(n_1-n_2, n_2-n_3,\dots,n_{N-1}-n_N). \tag{6.13}$$

Corollary 6.4. *If $\vec{\boldsymbol{x}}$ is d-invariant, then $T\vec{\boldsymbol{x}}$ is parallel to $\vec{\boldsymbol{x}}$.*

A d-invariant lattice is characterized by the following property.

Proposition 6.5. *Let $\vec{\boldsymbol{x}}:\mathbb{Z}^N\to\mathbb{R}^M$ be a d-invariant lattice, then*

$$\Delta_i\boldsymbol{X}_i = (T_iQ_{ii})\boldsymbol{X}_i - \sum_{\ell=1}^{N}Q_{i\ell}\boldsymbol{X}_\ell, \tag{6.14}$$

and consequently,

$$\Delta_iQ_{ij} + \tilde{\Delta}_jQ_{ij} - Q_{ij}(\Delta_iQ_{ii} - \tilde{\Delta}_jQ_{jj}) + \sum_{\ell=1,\ell\neq i,j}^{N}Q_{i\ell}Q_{\ell j} = 0. \tag{6.15}$$

Proof. Eqs. (6.14) and (6.15) are a straightforward consequence of Eqs. (6.9) and (6.10), respectively. □

Remark. *Formula* (6.14) *implies that the N-dimensional d-invariant lattice is effectively contained in an N-dimensional subspace of* $\mathbb{R}^M$, *therefore without loss of generality, we can put in this section* $N = M$.
We present now the characterization of d-invariant lattices in terms of hyperplane lattices.

Theorem 6.6. *If the quadrilateral lattice* $\vec{\boldsymbol{x}} : \mathbb{Z}^N \to \mathbb{R}^N$ *is* d*-invariant, then its rotation coefficients* Q_{ij} *are also the backward rotation coefficients of its complementary lattice*

$$\tilde{P}^*_{ij} = Q_{ij}, \quad i \neq j = 1, \dots, N. \tag{6.16}$$

Proof. If $\vec{\boldsymbol{x}}$ is quadrilateral, then comparison of the formula (6.14) with Eqs. (3.17) and (3.26) proves the statement. □

7. The Egorov lattice

Definition 7.1 ([33]). A quadrilateral lattice is a *Egorov lattice* iff the internal angles corresponding to the vertices $T_i\vec{\boldsymbol{x}}$ and $T_j\vec{\boldsymbol{x}}$ are right angles (see Fig. 6).

Since the opposite angles of the elementary quadrilaterals of the Egorov lattice sum up to the flat angle we have the following result.

Corollary 7.2 ([33]). *The Egorov lattice is circular.*

Remark. *The Egorov lattice constraint can be written algebraically in the form*

$$\boldsymbol{X}_i \cdot T_i\boldsymbol{X}_j = 0, \quad i \neq j, \tag{7.1}$$

which implies the circularity condition (5.2).

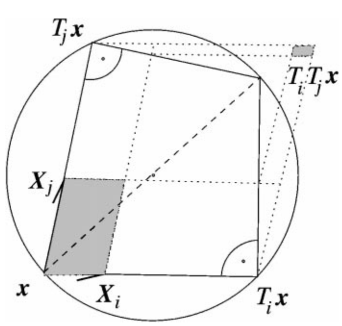

Fig. 6. Egorov lattice.

Corollary 7.3. *The line* $\langle \vec{\boldsymbol{x}}, T_i T_j \vec{\boldsymbol{x}} \rangle$ *is a main diagonal of the circle defined by the points* $\vec{\boldsymbol{x}}$, $T_i\vec{\boldsymbol{x}}$ and $T_j\vec{\boldsymbol{x}}$.

Proposition 7.4. *The Egorov lattice is an integrable reduction of the quadrilateral lattice.*

Proof. Define functions C^{E} by equation

$$C^{\mathrm{E}}_{ij} = \boldsymbol{X}_i \cdot T_i \boldsymbol{X}_j, \tag{7.2}$$

and notice the following identity:

$$T_k C^{\mathrm{E}}_{ij} = C^{\mathrm{E}}_{ij} + (T_k Q_{ik}) C^{\mathrm{E}}_{kj} + (T_i T_k Q_{jk}) C^{\mathrm{E}}_{ik} + (T_i T_k Q_{ji}) C^{\mathrm{E}}_{ki} \tag{7.3}$$

valid for a generic quadrilateral lattice. In the case of the Egorov lattice we have $C^{\mathrm{E}} = 0$, and Eq. (7.3) shows that such constraint is admissible. □

In the previous sections we introduced two other basic integrable reductions of the quadrilateral lattice: the symmetric and the d-invariant lattices. We will show that the Egorov lattice is symmetric and, for $N = M$, d-invariant.

Proposition 7.5. *The Egorov lattice is symmetric.*

Proof. The linear problem (1.4) and the constraint (7.1) imply that

$$\boldsymbol{X}_i \cdot \boldsymbol{X}_j + (T_i Q_{ji}) \boldsymbol{X}_i \cdot \boldsymbol{X}_i = 0, \quad i \neq j,$$

which gives

$$(T_j Q_{ij}) |\boldsymbol{X}_j|^2 = (T_i Q_{ji}) |\boldsymbol{X}_i|^2, \quad i \neq j. \tag{7.4}$$

Because the Egorov lattice is circular, then $|\boldsymbol{X}_i|^2$ can be identified with the potentials ρ_i, therefore Eq. (7.4) leads to the symmetry constraint (4.3). □

Remark. *An equivalent form of the constraint* (7.4) *was used by Schief* [35] *in his derivation of the Egorov lattice from the circular lattice*.

Remark. *The symmetry and circularity constraints are not enough to obtain algebraically the Egorov lattice. Indeed, consider a symmetric and circular lattice together with its tangent vectors* $\boldsymbol{X}_i$ *and the corresponding rotation coefficients* Q_{ij}*. The symmetry condition implies the existence of a* τ*-function* (*we call it* τ^{S}) *such that the potentials* $\rho^{\mathrm{S}}_i = T_i \tau^{\mathrm{S}} / \tau^{\mathrm{S}}$ *satisfy*

$$\rho^{\mathrm{S}}_i T_i Q_{ji} = \rho^{\mathrm{S}}_j T_j Q_{ij}. \tag{7.5}$$

The circularity condition, in turn, implies existence of a τ*-function* (*we call it* τ^{C}) *such that the corresponding potentials* ρ^{C}_i *are given by*

$$\rho^{\mathrm{C}}_i = |\boldsymbol{X}_i|^2. \tag{7.6}$$

Eqs. (2.17)–(2.19) *imply that the potentials* ρ_i^{C} and ρ_i^{S} *are connected by functions of single variables*

$$\rho_i^{\mathrm{C}}(n) = a_i(n_i)\rho_i^{\mathrm{S}}(n), \quad i = 1, \dots, N. \tag{7.7}$$

The Egorov lattice corresponds to the distinguished case in which we have $a_i \equiv 1$, $i = 1, \dots, N$.

Corollary 7.6. *In the circular lattice* $|T_i\tilde{\boldsymbol{X}}_i| = 1/|\boldsymbol{X}_i|$, *which implies that the parallelogram* $P(\boldsymbol{X}_i, \boldsymbol{X}_j)$ *is anti-similar to the parallelogram* $P(T_i\tilde{\boldsymbol{X}}_i, T_j\tilde{\boldsymbol{X}}_j)$. *In the Egorov lattice the parallelogram* $P(\boldsymbol{X}_i, \boldsymbol{X}_j)$ *is also anti-similar to the parallelogram* $P(\Delta_i\boldsymbol{X}_j, \Delta_j\boldsymbol{X}_i)$.

For $N = M$ the Egorov lattice exhibits the d-invariance property [34].

Proposition 7.7. *The Egorov lattice* $\vec{\boldsymbol{x}} : \mathbb{Z}^N \to \mathbb{R}^N$ *is* d*-invariant*.

Proof. The orthogonality conditions (7.1) imply that

$$\boldsymbol{X}_i \perp \langle T_i\boldsymbol{X}_\ell\rangle_{\ell=1,\ell\neq i}^N, \tag{7.8}$$

$$T\boldsymbol{X}_i \perp \langle TT_\ell^{-1}\boldsymbol{X}_\ell\rangle_{\ell=1,\ell\neq i}^N, \tag{7.9}$$

where $\langle T_i\boldsymbol{X}_\ell\rangle_{\ell=1,\ell\neq i}^N$ is the linear space spanned by $\{T_i\boldsymbol{X}_\ell\}_{\ell=1}^N$, $\ell \neq i$. In addition, the planarity of the lattice implies that these two linear subspaces coincide, therefore, $\boldsymbol{X}_i$ and $T\boldsymbol{X}_i$, which are orthogonal to the same $(N-1)$-dimensional linear subspace, must be proportional:

$$T\boldsymbol{X}_i = a_i\boldsymbol{X}_i. \tag{7.10}$$

Applying T to the linear system (1.4) and using (7.10), we infer that $a_i = a_i(n_i)$ $(= 1$ without loss of generosity) and $TQ_{ij} = Q_{ij}$. □

We conclude this section considering the Egorov lattice from the point of view of the parallel system $\vec{\boldsymbol{x}}_{(k)}$ and of its connections with hyperplane lattices. The results are a straightforward consequence of Propositions 4.9 and 5.6 and of the definition of the Egorov lattice.

Proposition 7.8. *Given a system of parallel quadrilateral lattices* $\{\vec{\boldsymbol{x}}_{(k)}\}$, $k = 1, \dots, M$, *and the associated matrix* $\boldsymbol{\Omega}$, *the following properties are equivalent.*

1. *The matrix* $\boldsymbol{\Omega}/R$ *is symmetric and orthogonal:*

$$\boldsymbol{\Omega}^{\mathrm{T}} = R^2\boldsymbol{\Omega}^{-1} = \boldsymbol{\Omega} \Rightarrow \boldsymbol{\Omega}^2 = R^2 I. \tag{7.11}$$

2. *The polar hyperplane lattice* $\mathcal{P}(\vec{\boldsymbol{x}}_{(k)})$ *coincides with the dual hyperplane lattice* $R^2\vec{\boldsymbol{y}}_{(k)}^*$ *and with the adjoint hyperplane lattice:*

$$\mathcal{P}(\vec{\boldsymbol{x}}_{(k)}) = \vec{\boldsymbol{x}}_{(k)} = R^2\vec{\boldsymbol{y}}_{(k)}^*, \quad k = 1, \dots, M. \tag{7.12}$$

The continuous limit of Eqs. (1.5), (4.2) and (5.2), namely Eqs. (1.10), (4.29) and (5.26), respectively, characterize submanifolds parametrized by Egorov systems of conjugate coordinates (Egorov nets). Also, the continuous limit of (6.15) together with (4.29) leads to the Lamé equations

$$\partial_i\beta_{ij} + \partial_j\beta_{ji} + \sum_{\ell=1,\ell\neq i,j}^{N} \beta_{i\ell}\beta_{j\ell} = 0, \tag{7.13}$$

which together with Eqs. (1.10) and (4.29), provide the usual characterization of a Egorov net. At last, the d-invariance properties (6.11) and (6.12) reduce to

$$\sum_{\ell=1}^{N}\partial_\ell\beta_{ij} = 0, \tag{7.14}$$

$$\sum_{\ell=1}^{N}\partial_\ell\boldsymbol{X}_i = 0, \tag{7.15}$$

implying that $\beta_{ij} = \beta_{ij}(u_1 - u_2, \ldots, u_{N-1} - u_N)$. For $N = 3$, we recover a classical characterization of the Egorov net [2,9].

8. $\bar{\partial}$ formulations of the reduction

In this section we prove that the distinguished reductions of the quadrilateral lattice discussed in the previous sections are integrable via the $\bar{\partial}$ reduction method introduced in [40] and generalized to a discrete context in [16]. For the sake of completeness, we first summarize in Sections 8.1 and 8.2, the $\bar{\partial}$ formulation of the quadrilateral lattice and the main result of the $\bar{\partial}$ reduction theory applied to it.

The $\bar{\partial}$ dressing method is a very convenient tool to construct integrable multidimensional systems, together with large classes of solutions [7,38,39]. Consider the (by assumption, uniquely solvable) matrix $M \times M$ $\bar{\partial}$ problem

$$\partial_{\bar{\lambda}}\phi(\lambda) = \partial_{\bar{\lambda}}\eta(\lambda) + \int_{\mathbb{C}} R(\lambda,\lambda')\phi(\lambda')\,\mathrm{d}\lambda'\wedge\mathrm{d}\bar{\lambda}', \quad \lambda,\lambda'\in\mathbb{C}, \tag{8.1}$$

where $\partial_{\bar{\lambda}} = \partial/\partial\bar{\lambda}$, the given rational function $\eta(\lambda)$ (the normalization of $\phi(\lambda)$) describes the singularities and the asymptotic behavior of ϕ in the complex plane and $R(\lambda,\lambda')$ is the given $M \times M$ matrix $\bar{\partial}$-datum; consider also the adjoint $\bar{\partial}$ problem:

$$\partial_{\bar{\lambda}}\phi^*(\lambda) = -\partial_{\bar{\lambda}}\eta(\lambda) - \int_{\mathbb{C}} \phi^*(\lambda')R(\lambda',\lambda)\,\mathrm{d}\lambda'\wedge\mathrm{d}\bar{\lambda}', \quad \lambda,\lambda'\in\mathbb{C}. \tag{8.2}$$

The above $\bar{\partial}$ problems imply the bilinear identity

$$\int_{C_\infty}\phi_2^*(\lambda)\phi_1(\lambda)\,\mathrm{d}\lambda + \int_{\mathbb{C}}[\phi_2^*(\lambda)\partial_{\bar{\lambda}}\eta_1(\lambda) - (\partial_{\bar{\lambda}}\eta_2(\lambda))\phi_1(\lambda)]\,\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda} = 0 \tag{8.3}$$

(where C_∞ is the circle with center at the origin and arbitrarily large radius, and the corresponding integration is counter-clockwise), which involves the solutions ϕ_1 and ϕ_2^* of (8.1) and (8.2) corresponding to the normalizations η_1 and η_2, respectively.

The dependence of the $M \times M$ matrices ϕ, ϕ^* and R on $\bar{\lambda}$ and $\bar{\lambda}'$: $\phi = \phi(\lambda, \bar{\lambda})$, $R = R(\lambda, \bar{\lambda}, \lambda', \bar{\lambda}')$ will be omitted systematically throughout the paper.

In the following, we shall consider only the two basic solutions $\chi(\lambda)$ and $\chi(\lambda, \mu)$ of Eq. (8.1), corresponding, respectively, to the “canonical normalization” $\eta = 1$ and to the “simple pole normalization” $\eta = (\lambda - \mu)^{-1}$ [22,23], together with the corresponding solutions of the adjoint problem (8.2) $\chi^*(\lambda)$ and $\chi^*(\lambda, \mu)$.

8.1. $\bar{\partial}$ formulation of the quadrilateral lattice

It turns out that the MQL equations are integrable via the $\bar{\partial}$-dressing method [5,16] and all the geometric quantities of the lattice have a distinguished role in this $\bar{\partial}$ scheme.

Proposition 8.1. *Let the $M \times M$ $\bar{\partial}$-datum R depend on the lattice variable $n = (n_1, \dots, n_N) \in Z^N$ in the following way:*

$$R(n; \lambda, \lambda') = (g(n, \lambda))^{-1} R_0(\lambda, \lambda') g(n, \lambda), \tag{8.4}$$

$$g(n, \lambda) = \prod_{k=1}^{N} [I + (\lambda - 1) P_k]^{n_k}, \tag{8.5}$$

where $R_0(\lambda, \lambda')$ is an arbitrary function of λ and λ', but constant in n and P_i, $i = 1, \dots, N$, are the usual ith projection matrices: $(P_i)_{jk} = \delta_{ij}\delta_{ik}$. Then the following results hold.

1. *The matrix functions*

$$\psi(\lambda) := g(n; \lambda)\chi(\lambda), \qquad \psi^*(\lambda) := \chi^*(\lambda)(g(n; \lambda))^{-1} \tag{8.6}$$

satisfy the following linear systems

$$\Delta_i \psi_{kj}(\lambda) = (T_i Q_{ji})\psi_{ki}(\lambda), \quad i = 1, \dots, N, \quad j, k = 1, \dots, M, \quad i \neq j, \tag{8.7}$$

$$\Delta_i \psi^*_{jk}(\lambda) = (T_i \psi^*_{ik}(\lambda)) Q_{ij}, \quad i = 1, \dots, N, \quad j, k = 1, \dots, M, \quad i \neq j, \tag{8.8}$$

respectively, where Q_{ij} is the (ij)-component of the matrix Q defined by

$$Q = \lim_{\lambda \to \infty} (\chi^{\mathrm{T}}(\lambda) - I) = \lim_{\lambda \to \infty} (I - \lambda(\chi^{*\mathrm{T}}(\lambda)). \tag{8.9}$$

2. *The matrix function*

$$\psi(\lambda, \mu) := g(n; \lambda)\chi(\lambda, \mu)(g(n; \mu))^{-1} \tag{8.10}$$

is connected to the canonically normalized solutions of the $\bar{\partial}$ problem through the equations

$$\Delta_i \psi_{kj}(\lambda, \mu) = \psi_{ki}(\lambda) T_i \psi^*_{ij}(\mu), \quad i = 1, \dots, N, \quad j, k = 1, \dots, M. \tag{8.11}$$

Furthermore, the matrix function

$$\psi^*(\lambda,\mu) := g(n;\mu)\chi^*(\lambda,\mu)(g(n,\lambda))^{-1} \tag{8.12}$$

is connected to $\psi(\lambda,\mu)$ *via*

$$\psi^*(\mu,\lambda) = \psi(\lambda,\mu) \tag{8.13}$$

and the canonically normalized solutions of the $\bar\partial$ *problem can be obtained from* $\chi(\lambda,\mu)$ *via the asymptotics* [6]:

$$\chi^*(\mu) = \lim_{\lambda\to\infty}[\lambda\chi(\lambda,\mu)], \qquad \chi(\lambda) = -\lim_{\mu\to\infty}[\mu\chi(\lambda,\mu)] \tag{8.14}$$

and

$$T_i\chi_{ji}(\lambda,0) = \chi_{ji}(\lambda)T_i\chi^*_{ii}(0), \qquad \chi_{ij}(0,\mu) = -\chi_{ii}(0)T_i\chi^*_{ij}(\mu). \tag{8.15}$$

Proof. The proof is standard in the philosophy of the $\bar\partial$ method.

1. First, after defining the "long derivatives"

$$(\mathcal{D}_i f)(\lambda) = \Delta_i f + (\lambda-1)P_iT_i f, \qquad (\mathcal{D}^*_i f)(\lambda) = -\tilde\Delta_i f + (\lambda-1)(T_i^{-1}f)P_i,$$

one can verify that the functions

$$(\mathcal{D}_i\chi)(\lambda)P_j - \chi(\lambda)P_i(T_iQ^{\mathrm{T}})P_j,$$
$$P_j(\mathcal{D}^*_i\chi^*)(\lambda) - P_j(T_i^{-1}Q^{*\mathrm{T}})P_i\chi^*(\lambda), \qquad i\neq j,$$

where Q^*_{ij} is the (ij)-component of the matrix Q^* defined by

$$Q^* = \lim_{\lambda\to\infty}(\chi^{*\mathrm{T}}(\lambda) - I), \tag{8.16}$$

solve the homogeneous version of the $\bar\partial$ problems (8.1) and (8.2) and go to zero at $\lambda\to\infty$; therefore, uniqueness implies the equations

$$(\mathcal{D}_i\chi)(\lambda)P_j = \chi(\lambda)P_i(T_iQ^{\mathrm{T}})P_j, \quad i\neq j, \tag{8.17}$$

$$P_j(\mathcal{D}^*_i\chi^*)(\lambda) = P_j(T_i^{-1}Q^{*\mathrm{T}})P_i\chi^*(\lambda), \quad i\neq j, \tag{8.18}$$

or, equivalently, the equations

$$\Delta_i\psi(\lambda)P_j = \psi(\lambda)P_i(T_iQ^{\mathrm{T}})P_j, \quad i\neq j, \tag{8.19}$$

$$P_j\Delta_i\psi^*(\lambda) = -P_j(T_i^{-1}Q^{*\mathrm{T}})P_iT_i\psi^*(\lambda), \quad i\neq j. \tag{8.20}$$

These last two equations, written in components, coincide with (8.7) and (8.8), using also the property

$$Q^* = -Q, \tag{8.21}$$

which is a direct consequence of the bilinear identity (8.3) for $\chi(\lambda)$ and $\chi^*(\lambda)$. At last, the $\lambda\to\infty$ limit of Eq. (8.17) implies that the coefficients Q_{ij} satisfy the MQL equations (1.5).

2. The proof of formulas (8.11) is conceptually similar. The function

$$\mathcal{D}_i(\chi(\lambda,\mu)(g(\mu))^{-1}) - \chi(\lambda)P_i T_i \varphi(\mu), \tag{8.22}$$

where

$$\varphi(\mu) = \lim_{\lambda\to\infty} \lambda\chi(\lambda,\mu)(g(\mu))^{-1} \tag{8.23}$$

solves the homogeneous version of the $\bar\partial$ problem (8.1) and goes to zero at $\lambda \to \infty$, therefore uniqueness implies the equation

$$\mathcal{D}_i(\chi(\lambda,\mu)(g(\mu))^{-1}) = \chi(\lambda)P_i T_i \varphi(\mu). \tag{8.24}$$

This equation is equivalent to

$$\Delta_i \psi(\lambda,\mu) = \psi(\lambda)P_i T_i \varphi(\mu), \tag{8.25}$$

whose component form reduces to (8.11), taking account of the formulas

$$\varphi(\mu) = \psi^*(\mu), \qquad \varphi^*(\mu) = -\psi(\mu), \tag{8.26}$$

which are obtained from the bilinear identity (8.3) for $\chi(\lambda,\mu),\ \chi^*(\lambda)$ and $\chi^*(\lambda,\mu)$, $\chi(\lambda)$, respectively. At last, the bilinear identity (8.3) for $\eta_1 = (\lambda-\mu)^{-1},\ \eta_2 = (\lambda-\mu')^{-1}$ gives $\chi^*(\mu',\mu) = \chi(\mu,\mu')$ or, equivalently, Eq. (8.13); furthermore, Eqs. (8.13), (8.23) and (8.26) lead to Eqs. (8.14) and (8.24), which when evaluated at $\lambda = 0$, gives Eq. (8.15). □

From the solutions $\psi(\lambda,\mu),\ \psi(\lambda)$ and $\psi^*(\lambda)$ of the $\bar\partial$ problem one can construct a system $\{\vec{\boldsymbol{x}}_{(k)}\},\ k = 1,\dots,M$, of parallel quadrilateral lattices, together with the corresponding tangent vectors and Lamé coefficients through the following matrix equations:

$$\boldsymbol{\Omega} = \int_{\mathbb{C}} \mathrm{d}\lambda \wedge \mathrm{d}\bar\lambda \int_{\mathbb{C}} \mathrm{d}\mu \wedge \mathrm{d}\bar\mu\, M(\lambda)\psi(\lambda,\mu)M^*(\mu), \tag{8.27}$$

$$\boldsymbol{X}_i = \int_{\mathbb{C}} \mathrm{d}\lambda \wedge \mathrm{d}\bar\lambda\, M(\lambda)\psi_i(\lambda), \qquad \boldsymbol{X}_i^* = \int_{\mathbb{C}} \mathrm{d}\mu \wedge \mathrm{d}\bar\mu\, \psi_i^*(\mu)M^*(\mu), \tag{8.28}$$

where $\vec{\boldsymbol{x}}_{(i)}$ is the ith column of matrix $\boldsymbol{\Omega}$, $\psi_i(\lambda)$ is the ith column of matrix $\psi(\lambda)$, $\psi_i^*(\mu)$ is the ith row of matrix $\psi^*(\mu)$, and $M(\lambda)$ and $M^*(\lambda)$ are arbitrary $M\times M$ matrices independent of n.

Finally, the evaluation of Eq. (8.17) at the distinguished point $\lambda = 0$ leads to the τ-function representation (2.15) and (2.16) of the MQL lattice. Indeed, at $\lambda = 0$, Eq. (8.17) reads

$$\Delta_i \chi_{jj}(0) = \chi_{ji}(0)T_i Q_{ji}, \tag{8.29}$$

$$\chi_{ij}(0) + \chi_{ii}(0)T_i Q_{ji} = 0, \tag{8.30}$$

and imply that

$$\frac{\Delta_i \chi_{jj}(0)}{\chi_{jj}(0)} = -(T_i Q_{ji})(T_j Q_{ij}). \tag{8.31}$$

Comparing Eq. (8.31) with Eq. (2.8) leads to the identification

$$\chi_{ii}(0) = \rho_i = \frac{T_i \tau}{\tau}, \tag{8.32}$$

while Eq. (8.30) gives

$$\chi_{ji}(0) = -\frac{T_j \tau_{ij}}{\tau}, \quad i \neq j. \tag{8.33}$$

It is also possible to express $\chi^*_{ii}(0)$ and $\chi^*_{ij}(0)$ in terms of τ and τ_{ij}. To do so, we remark that the function $\phi_2(\lambda) = T_i \chi^*(\lambda)(I + (\lambda - 1)P_i)^{-1}$ satisfies Eq. (8.2) corresponding to the forcing $\pi\delta(\lambda)P_i T_i \chi^*(0)$. The bilinear equation (8.3) with this ϕ_2 and with $\phi_1(\lambda) = \chi(\lambda)$ reduces to the following equation:

$$T_i Q^{\mathrm{T}}(I - P_i) + P_i + (I - P_i)Q^{\mathrm{T}} = (T_i \chi^*(0))P_i \chi(0), \tag{8.34}$$

whose ii and (ij)-components read as

$$(T_i \chi^*_{ii}(0))\chi_{ii}(0) = 1, \qquad (T_i \chi^*_{ji}(0))\chi_{ii}(0) = Q_{ij}, \tag{8.35}$$

implying that

$$\chi^*_{ii}(0) = \frac{1}{T_i^{-1}\rho_i} = \frac{T_i^{-1}\tau}{\tau}, \qquad \chi^*_{ji}(0) = \frac{T_i^{-1}\tau_{ij}}{\tau}. \tag{8.36}$$

8.2. $\bar{\partial}$-reduction theory of the quadrilateral lattice

The above $\bar{\partial}$ formulation allows one to look for reductions of the MQL at the simpler level of the $\bar{\partial}$-datum R [16]. The particular form (8.4) of it implies the following proposition.

Proposition 8.2. *The following linear constraint on the $\bar{\partial}$-datum $R(\lambda, \lambda')$:*

$$R^{\mathrm{T}}(\lambda^{-1}, \lambda'^{-1}) = |\lambda'|^4 \bar{\lambda}^2 F(\lambda') R(\lambda', \lambda)(F(\lambda))^{-1} \tag{8.37}$$

gives rise to integrable reductions of the MQL. In formula (8.37),

$$F_{\pm}(\lambda) = \lambda^{-1}[A(\lambda) \pm A(\lambda^{-1})] \tag{8.38}$$

and $A(\lambda)$ is an arbitrary diagonal matrix.

The main implication of the constraint (8.37) is that the function $\phi^{\mathrm{T}}(\lambda^{-1})F(\lambda)$ satisfies the adjoint $\bar{\partial}$ problem (8.2), while the function $F^{-1}(\lambda^{-1})\phi^{*\mathrm{T}}(\lambda^{-1})$ satisfies the $\bar{\partial}$ problem (8.1):

$$\begin{aligned}\partial_{\bar{\lambda}}(\phi^{\mathrm{T}}(\lambda^{-1})F(\lambda)) = {} & \phi^{\mathrm{T}}(\lambda^{-1})\partial_{\bar{\lambda}}F(\lambda) + (\partial_{\bar{\lambda}}\eta(\lambda^{-1}))F(\lambda) \\ & - \int_{\mathbb{C}} (\phi^{\mathrm{T}}(\lambda'^{-1})F(\lambda'))R(\lambda', \lambda)\, \mathrm{d}\lambda' \wedge \mathrm{d}\bar{\lambda}',\end{aligned} \tag{8.39}$$

$$\partial_{\bar\lambda}(F^{-1}(\lambda^{-1})\phi^{*\mathrm{T}}(\lambda^{-1})) = (\partial_{\bar\lambda}F^{-1}(\lambda^{-1}))\phi^{*\mathrm{T}}(\lambda^{-1}) - F^{-1}(\lambda^{-1})\partial_{\bar\lambda}\eta(\lambda^{-1}) + \int_{\mathbb{C}} R(\lambda,\lambda')(F^{-1}(\lambda'^{-1})\phi^{*\mathrm{T}}(\lambda'^{-1}))\,\mathrm{d}\lambda'\wedge \mathrm{d}\bar\lambda', \tag{8.40}$$

and these equations, through the bilinear identity (8.3), imply the non-local quadratic constraints:

$$\int_{C_\infty}\phi^{\mathrm{T}}(\lambda^{-1})F(\lambda)\phi(\lambda)\,\mathrm{d}\lambda + \int_{\mathbb{C}}[\phi^{\mathrm{T}}(\lambda^{-1})(\partial_{\bar\lambda}F(\lambda))\phi(\lambda) + (\partial_{\bar\lambda}\eta(\lambda^{-1}))F(\lambda)\phi(\lambda) + \phi^{\mathrm{T}}(\lambda^{-1})F(\lambda)\partial_{\bar\lambda}\eta(\lambda)]\,\mathrm{d}\lambda\wedge\mathrm{d}\bar\lambda = 0, \tag{8.41}$$

$$\int_{C_\infty}\phi^{*}(\lambda)F^{-1}(\lambda^{-1})\phi^{*\mathrm{T}}(\lambda^{-1})\,\mathrm{d}\lambda + \int_{\mathbb{C}}[\phi^{*}(\lambda)(\partial_{\bar\lambda}F^{-1}(\lambda^{-1}))\phi^{*\mathrm{T}}(\lambda^{-1}) - (\partial_{\bar\lambda}\eta(\lambda))F^{-1}(\lambda^{-1})\phi^{*\mathrm{T}}(\lambda^{-1}) - \phi^{*}(\lambda)F^{-1}(\lambda^{-1})\partial_{\bar\lambda}\eta(\lambda^{-1})]\,\mathrm{d}\lambda\wedge\mathrm{d}\bar\lambda = 0. \tag{8.42}$$

Therefore, the constraint (8.37) establishes a non-trivial connection, whose nature depends on the particular choice of $F(\lambda)$ (or, better, of $A(\lambda)$), between the solutions of the $\bar\partial$ problem (8.1) and of its adjoint (8.2) or, equivalently, between quadrilateral lattices and their dual objects, the quadrilateral hyperplane lattices. In the following, we shall identify the matrix functions $A(\lambda)$ which correspond to the symmetric, circular and Egorov lattices.

8.3. $\bar\partial$ formulation of the symmetric lattice

In this section we solve the symmetric lattice. We shall show that the following choice:

$$A(\lambda) = \frac{I}{2} \Rightarrow F_{+}(\lambda) = \lambda^{-1}I \tag{8.43}$$

corresponds to the symmetric lattice reduction.

Proposition 8.3. *Let $F(\lambda)=\lambda^{-1}I$, then the following equations hold:*

$$\psi^{\mathrm{T}}(\lambda,\mu) = (\lambda\mu)^{-1}\psi^{*}(\mu^{-1},\lambda^{-1}), \tag{8.44}$$

$$\lambda^{-1}\psi_{ji}(\lambda^{-1}) = \frac{T_i\tau}{\tau}\psi^{*}_{ij}(\lambda), \tag{8.45}$$

$$\chi^{\mathrm{T}}(0) = \chi(0), \tag{8.46}$$

and Eqs. (8.27) and (8.28) *allow to construct a system of symmetric lattices provided that*

$$M^{*}(\lambda) = \lambda|\lambda|^{-4}M^{\mathrm{T}}(\lambda). \tag{8.47}$$

Proof. We use the same strategy of the previous $\bar\partial$ proofs. Comparing Eq. (8.39) with Eq. (8.2) for $\eta = (\lambda-\mu)^{-1}$, one obtains

$$\chi^{\mathrm{T}}(\lambda^{-1},\mu^{-1}) = \lambda\mu\chi^{*}(\lambda,\mu), \tag{8.48}$$

or equivalently, (8.44), using Eqs. (8.11) and (8.13). Furthermore, one can verify that $T_i\chi^*(\lambda)(I-(\lambda^{-1}-1)P_i)$ satisfies the $\bar\partial$ equation (8.1) for $\eta = T_i\chi^*(0)P_i$. Therefore, taking account of the λ large asymptotics, one obtains the equation

$$T_i\chi^*(\lambda)(I-(\lambda^{-1}-1)P_i) - \lambda^{-1}(T_i\chi^*(0))P_i\chi^{\mathrm{T}}(\lambda^{-1}) = (I-P_i)\chi^*(\lambda), \tag{8.49}$$

whose (ij)-component gives (8.45), using Eqs. (8.6) and (8.36). At last, Eq. (8.41) for $\eta = 1$ gives directly (8.46), which can be immediately identified with the symmetry constraint (4.3), using Eqs. (2.10), (8.32) and (8.33). Furthermore, Eqs. (8.44) and (8.45) imply Eqs. (4.24) and (4.26), provided that one uses (8.47). □

8.4. $\bar\partial$ formulation of the circular lattice

It was shown in [16] that the following choice

$$A(\lambda) = (\lambda-1)^{-1}I \Rightarrow F_-(\lambda) = \frac{\lambda+1}{\lambda(\lambda-1)}I \tag{8.50}$$

corresponds to the circular lattice reduction.

Proposition 8.4. *Let $F(\lambda) = ((\lambda+1)/\lambda(\lambda-1))I$, then the following equations hold:*

$$\chi(0) + \chi^{\mathrm{T}}(0) = 2\chi^{\mathrm{T}}(1)\chi(1), \tag{8.51}$$

$$\chi^*(0) + \chi^{*\mathrm{T}}(0) = 2\chi^*(-1)\chi^{*\mathrm{T}}(-1), \tag{8.52}$$

$$\frac{\lambda+1}{\lambda(1-\lambda)}\chi^{\mathrm{T}}(\lambda^{-1},\mu^{-1}) = \frac{\mu(\mu+1)}{1-\mu}\chi(\mu,\lambda) + \chi^{\mathrm{T}}(1,\mu^{-1})\chi(1,\lambda), \tag{8.53}$$

$$\frac{\lambda-1}{\lambda(1+\lambda)}\chi^{\mathrm{T}}(\mu^{-1},\lambda^{-1}) = \frac{\mu(\mu-1)}{1+\mu}\chi(\lambda,\mu) + \chi(\lambda,-1)\chi^{\mathrm{T}}(\mu^{-1},-1), \tag{8.54}$$

$$4\chi^{\mathrm{T}}(1,-1)\chi(1,-1) = I. \tag{8.55}$$

Proof. Eqs. (8.41) and (8.42) for $\eta = 1$ give, respectively, Eqs. (8.51) and (8.52). Consider Eq. (8.39) for $\eta = (\lambda-\mu)^{-1}$, then Eq. (8.53) follows from the fact that its RHS satisfies Eq. (8.41) as well. Analogous considerations lead to Eq. (8.54). At last Eq. (8.53), evaluated at $\lambda = \mu = -1$, gives the orthogonality condition (8.55). □

To show that the above formulas give rise to a circular lattice, consider the following identification:

$$\vec{\boldsymbol{x}}_{(i)} = (\psi_{1i}(1,\mu),\ldots,\psi_{Mi}(1,\mu))^{\mathrm{T}}, \tag{8.56}$$

$$\boldsymbol{X}_i = (\psi_{1i}(1),\ldots,\psi_{Mi}(1))^{\mathrm{T}}, \qquad H_{i(n)} = \psi^*_{in}(\mu), \tag{8.57}$$

$$\boldsymbol{X}^*_i = (\psi^*_{i1}(-1),\ldots,\psi^*_{iM}(-1)). \tag{8.58}$$

Because of Eq. (8.32), the diagonal part of (8.51) leads to

$$\chi_{ii}(0) = \rho_i = |\boldsymbol{X}_i|^2, \tag{8.59}$$

while the off-diagonal part gives the circularity constraint (5.2). Evaluating Eq. (8.53) at $\mu = 0$ and using Eq. (8.14), one obtains

$$\frac{\lambda+1}{\lambda(\lambda-1)}\chi^{\mathrm{T}}(\lambda^{-1}) = \chi(0,\lambda) - 2\chi^{\mathrm{T}}(1)\chi(1,\lambda), \tag{8.60}$$

which, using Eqs. (8.15) and (8.59), can be written in the following form:

$$-\frac{\lambda+1}{\lambda(\lambda-1)}\psi(\lambda^{-1}) = (\vec{\boldsymbol{x}}_{(k)} + T_i\vec{\boldsymbol{x}}_{(k)})\cdot\boldsymbol{X}_k, \quad k = 1,\dots,M, \tag{8.61}$$

which is the $\bar{\partial}$ formulation of the first point of Proposition 5.5. If, instead, we choose $\mu = \lambda^{-1}$, we obtain

$$\frac{\lambda+1}{\lambda(1-\lambda)}[\psi(\lambda^{-1},\lambda) + \psi^{\mathrm{T}}(\lambda^{-1},\lambda)] = \psi^{\mathrm{T}}(1,\lambda)\psi(1,\lambda), \tag{8.62}$$

which, through the identification (8.56), leads to

$$\frac{\lambda+1}{\lambda(1-\lambda)}[\psi_{jk}(\lambda^{-1},\lambda) + \psi_{kj}(\lambda^{-1},\lambda)] = \boldsymbol{x}_{(j)}\cdot\boldsymbol{x}_{(k)}. \tag{8.63}$$

This formula states that the scalar product of the two parallel lattices $\vec{\boldsymbol{x}}_{(j)}, \vec{\boldsymbol{x}}_{(k)}, j \neq k$, such that

$$\Delta_i\vec{\boldsymbol{x}}_{(j)} = (T_iH_{i(j)})\boldsymbol{X}_i, \qquad H_{i(n)} = \psi^*_{in}(\mu) \tag{8.64}$$

is equal to the sum of two scalar solutions of the Laplace equations (1.1) and (1.7) corresponding, respectively, to the Lamé coefficients $H_{i(j)}, H_{i(k)}$. If $j = k$, Eq. (8.63) reduces to

$$|\vec{\boldsymbol{x}}_{(j)}|^2 = \frac{2(\lambda+1)}{\lambda(1-\lambda)}\psi_{jj}(\lambda^{-1},\lambda), \tag{8.65}$$

which is the $\bar{\partial}$ formulation of the second point of Proposition 8.44. Eq. (8.52) expresses the circularity condition (5.16) for hyperplane lattices through the identification (8.56) and Eq. (8.55) is the $\bar{\partial}$ formulation of Eq. (5.10), through the identification

$$\boldsymbol{\Omega} = \psi(1,-1), \quad R = 2. \tag{8.66}$$

In this case, both systems $\{\vec{\boldsymbol{x}}_{(k)}\}$ and $\{\vec{\boldsymbol{x}}^*_{(k)}\}$ are circular. We finally remark that Eqs. (8.53) and (8.54) contain all the other circular constraints for a suitable choice of λ and μ.

8.5. $\bar{\partial}$ formulation of the d-invariant lattice

The d-invariant lattice, a distinguished reduction of the quadrilateral lattice, corresponds to the following distributional $\bar{\partial}$-datum:

$$R(\lambda,\lambda') = \tfrac{1}{2}\mathrm{i}\delta(\lambda-\lambda')R(\lambda), \tag{8.67}$$

and is solved by the local $\bar\partial$ problem

$$\partial_{\bar\lambda}\chi(\lambda) = \chi(\lambda)R(\lambda), \tag{8.68}$$

$$T_i R(\lambda) = [1+(\lambda-1)P_i]R(\lambda)[1+(\lambda-1)P_i]^{-1}. \tag{8.69}$$

If $N = M$, from Eq. (8.69), the invariance property follows:

$$TR(\lambda) = R(\lambda), \tag{8.70}$$

which implies that

$$T\chi(\lambda) = \chi(\lambda). \tag{8.71}$$

Consequently,

$$T\psi(\lambda) = \lambda\psi(\lambda), \tag{8.72}$$

$$TQ = Q, \tag{8.73}$$

$$T\rho_i = \rho_i, \tag{8.74}$$

and taking $\lambda = 1$, we obtain formulae (6.11) and (6.12).

8.6. $\bar\partial$ formulation of the Egorov lattice

The Egorov lattice is circular and symmetric; therefore, the corresponding constraints are satisfied simultaneously, i.e.,

$$|\lambda'|^{-4}\bar\lambda^{-2}R^{\mathrm{T}}(\lambda'^{-1},\lambda^{-1}) = \lambda R(\lambda,\lambda')\lambda'^{-1} = \left(\frac{\lambda+1}{\lambda(\lambda-1)}\right)^{-1} R(\lambda,\lambda')\frac{\lambda'+1}{\lambda'(\lambda'-1)}. \tag{8.75}$$

This implies the equation

$$\frac{2\lambda(\lambda-\lambda')}{\lambda'(\lambda'-1)(\lambda+1)}R(\lambda,\lambda') = 0, \tag{8.76}$$

which admits the distributional solution (8.67). Therefore, the $\bar\partial$ formulation of the Egorov lattice is given in terms of the *local* $\bar\partial$ problem (8.68) and (8.69) in which the $\bar\partial$-datum satisfies the constraint

$$R^{\mathrm{T}}(\lambda^{-1}) = \bar\lambda^2 R(\lambda). \tag{8.77}$$

Because of this locality, the corresponding $\bar\partial$ reduction theory of Section 8.2 simplifies considerably.

The constraint (8.77) implies that $\chi^{\mathrm{T}}(\lambda^{-1})$ is a solution of the adjoint $\bar\partial$ problem

$$\partial_{\bar\lambda}\chi^*(\lambda) = -\chi^*(\lambda)R(\lambda), \tag{8.78}$$

and the corresponding quadratic constraint

$$\partial_{\bar{\lambda}}(\chi^{\mathrm{T}}(\lambda^{-1})\chi(\lambda)) = 0, \tag{8.79}$$

together with the asymptotics $\lim_{\lambda\to\infty}\chi^{\mathrm{T}}(\lambda^{-1})\chi(\lambda) = \chi^{\mathrm{T}}(0)$, imply that

$$\chi^{\mathrm{T}}(\lambda^{-1})\chi(\lambda) = \chi^{\mathrm{T}}(0). \tag{8.80}$$

Evaluating this constraint at $\lambda = 1$ and using the identifications (8.57), its diagonal part gives (8.59), while its off-diagonal part gives the Egorov constraint (7.1).

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